Bayesian Networks with Imprecise Probabilities: Theory and Applications to Knowledge-based Systems and Classification

A Tutorial by

Alessandro Antonucci, Giorgio Corani and Denis Mauá

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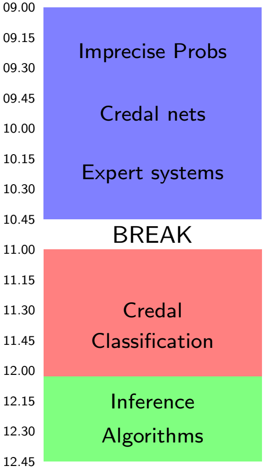
Istituto "Dalle Molle" di Studi sull'Intelligenza Artificiale - Lugano (Switzerland)

IJCAI-13

Beijing, August 5th, 2013

This morning

- From precise to imprecise probabilities
- Credal networks
- Applications to knowledge-based decision support systems
- Credal classifiers
- Application to data mining
- Inference algorithms on credal networks
- Discussion and general questions



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12.00
12.15
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12.45

Imprecise Probs

Credal nets

Expert systems

BREAK

Credal Classification

Inference Algorithms

Antonucci

Corani

giorgio

Mauá

denis

Just before starting . . .

Credal networks (i.e., Bayesian networks with imprecise probability) are drawing interest from the AI community in 2013

- ECSQARU 2013 Best Paper Award: *Approximating Credal Network Inferences by Linear Programming* by Alessandro Antonucci, Cassio de Campos, David Huber, and Marco Zaffalon
- UAI '13 Google Best Student Paper Award: *On the Complexity of Strong and Epistemic Credal Networks* by Denis Mauá, Cassio de Campos, Alessio Benavoli, and Alessandro Antonucci

More info and papers at `ipg.idsia.ch`

Outline (of the first part)

- A (first informal, then formal) introduction to IPs
    - Reasoning with (imprecise) fault trees
    - From determinism to imprecision (through uncertainty)
    - Motivations and coherence
- Credal sets
    - Basic concepts and operations
    - Modeling
- Credal networks
    - Background on Bayesian networks
    - From Bayesian to credal networks
    - Modeling (observations, missing data, information fusion, . . .)
- Applications to knowledge-based systems
    - Military decision making
    - Environmental risk analysis
    - (Imprecise) probabilistic description logic

Reasoning: from Determinism to IPs

brake fails $= [$ pads $\vee$ (sensor $\wedge$ controller $\wedge$ actuator) $]$

devices failures are independent

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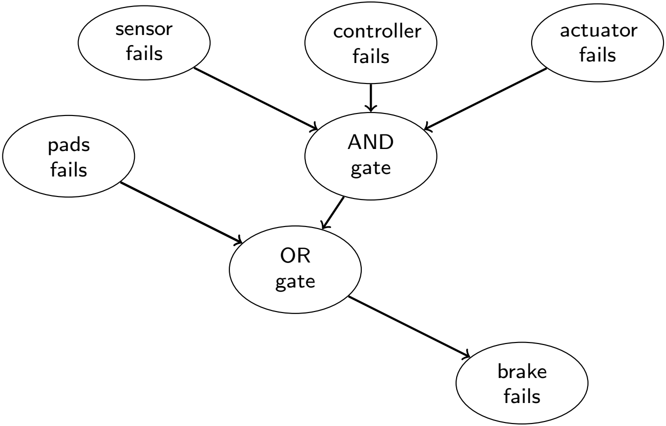

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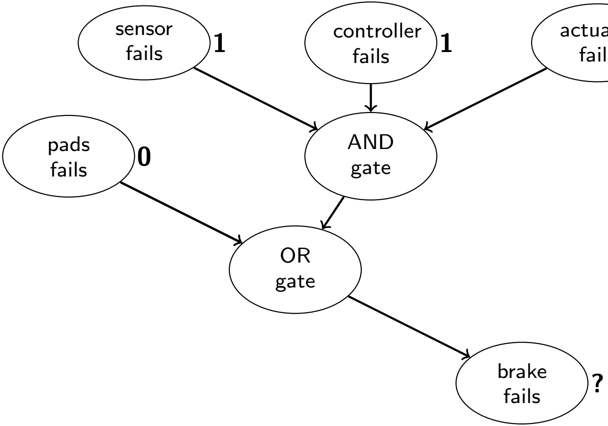

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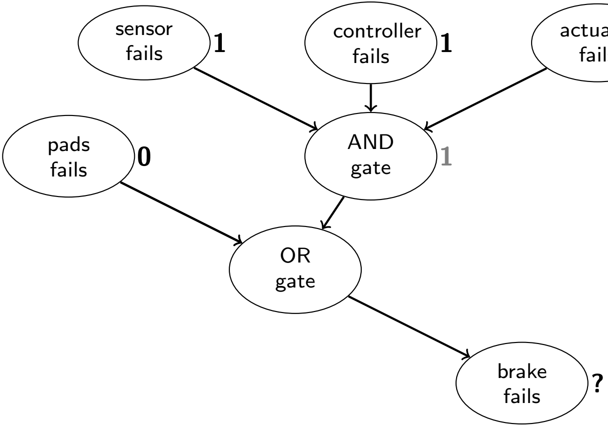

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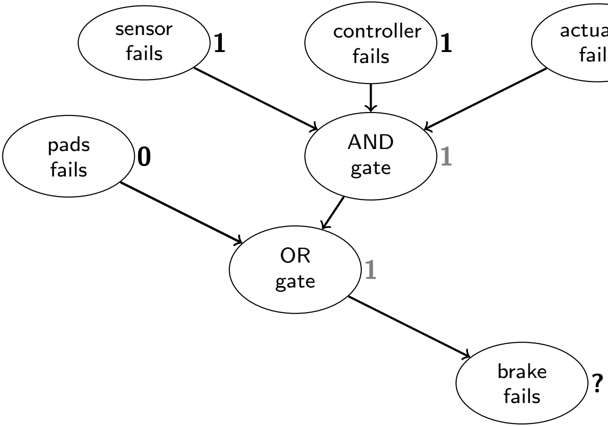

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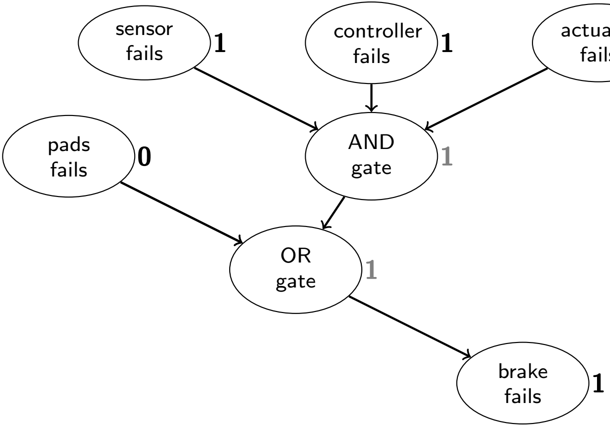

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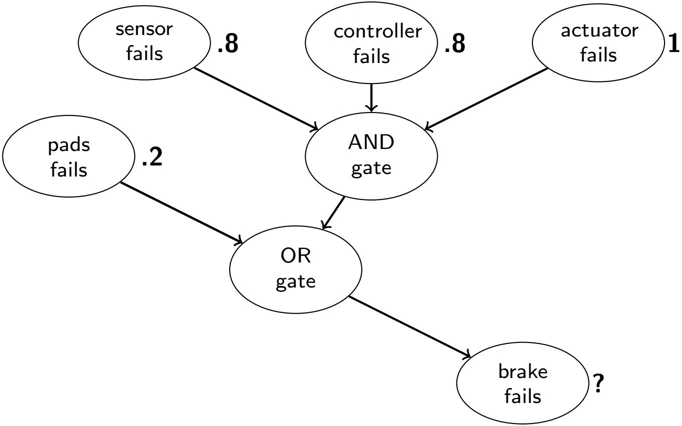

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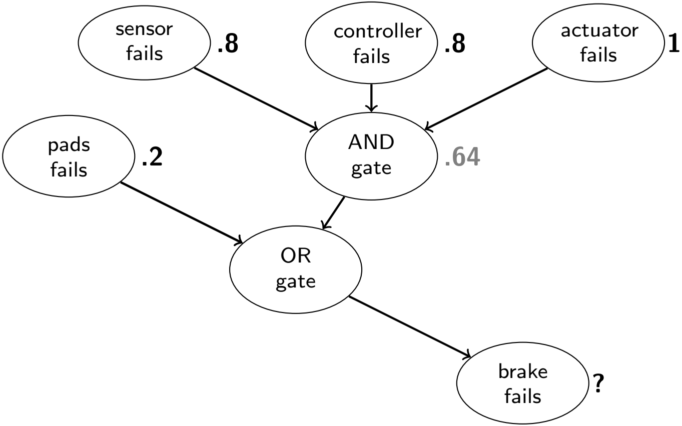

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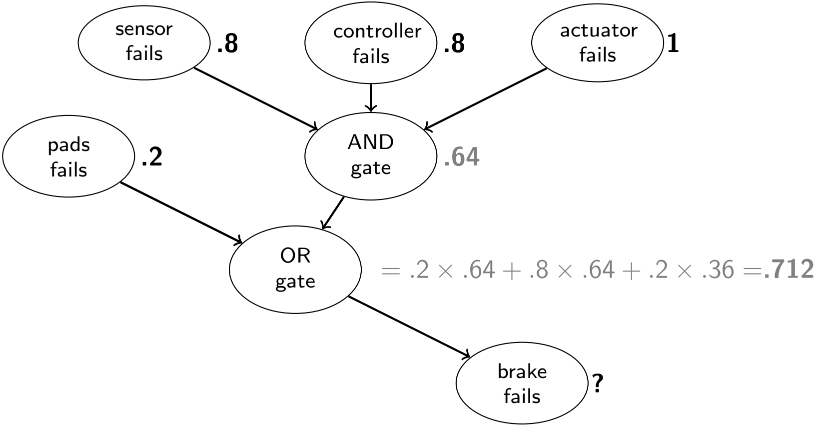

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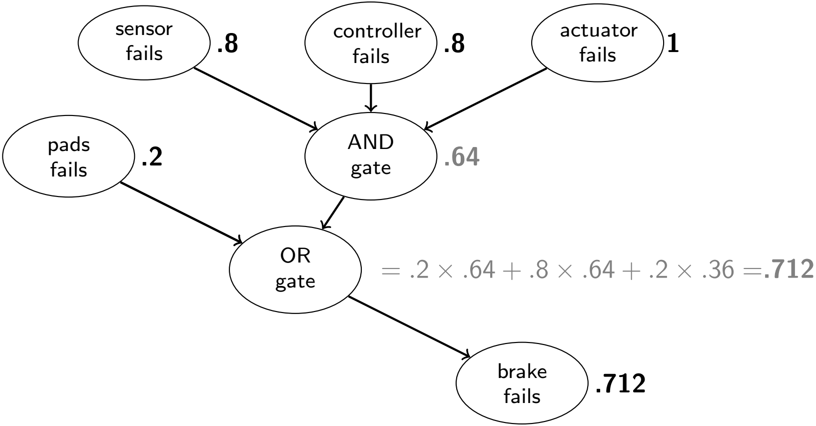

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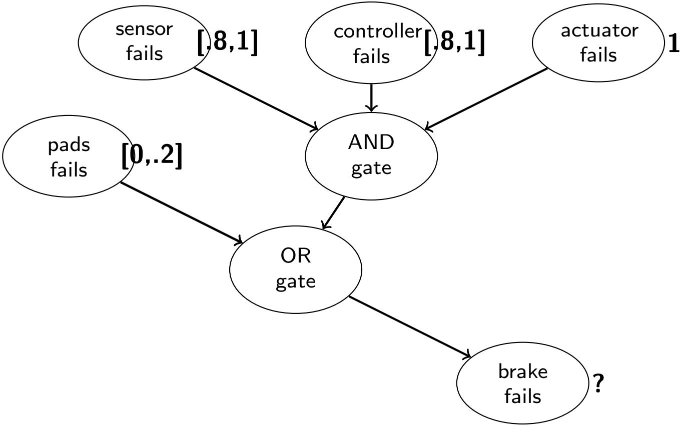

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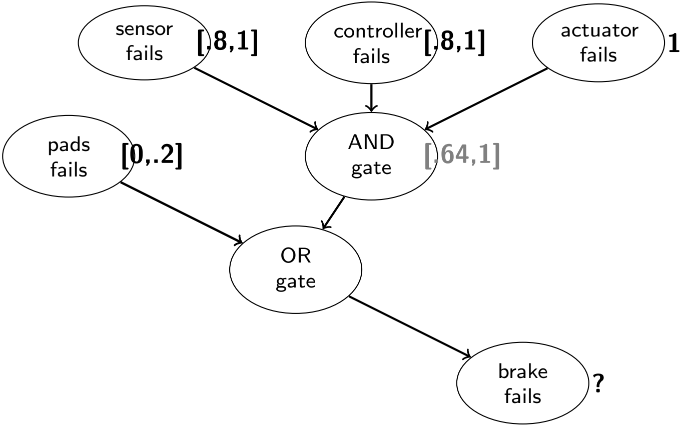

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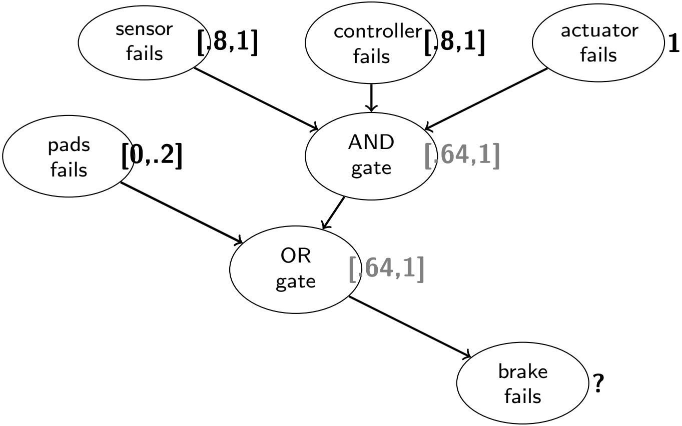

sensor fails **[.8,1]**

controller fails **[.8,1]**

actuator fails **1**

pads fails **[0,.2]**

AND gate **[.64,1]**

OR gate **[.64,1]**

brake fails **?**

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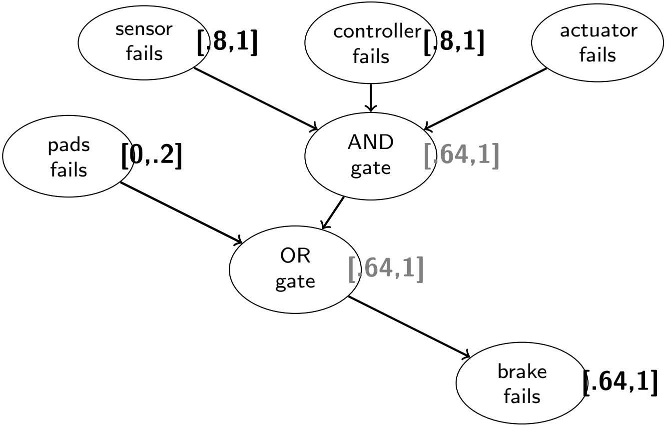

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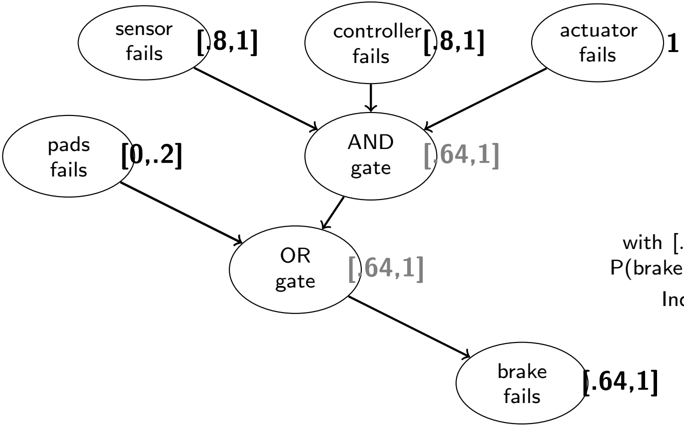

with $[.7, 1]$ instead
P(brake fails)∈[.49,1]
Indecision!

devices failures are independent

Three different levels of knowledge

- A football match between Italy and Spain
- Result of Spain after the regular time? Win, draw or loss?

DETERMINISM

The Spanish goalkeeper is unbeatable and Italy always receives a goal

Spain (certainly) wins

$$\begin{array}{l} P(Win) \\ P(Draw) \\ P(Loss) \end{array} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

UNCERTAINTY

Win is two times more probable than draw, and this being three times more probable than loss

$$\begin{array}{l} P(Win) \\ P(Draw) \\ P(Loss) \end{array} = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

IMPRECISION

Win is more probable than draw, and this is more probable than loss

$P(Win) > P(Draw)$
$P(Draw) > P(Loss)$

$$\begin{array}{l} P(Win) \\ P(Draw) \\ P(Loss) \end{array} = \begin{bmatrix} \frac{\alpha}{3} + \beta + \frac{\gamma}{2} \\ \frac{\alpha}{3} + \frac{\gamma}{2} \\ \frac{\alpha}{3} \end{bmatrix}$$

$\forall \alpha, \beta, \gamma$ such that
$\alpha > 0, \beta > 0, \gamma > 0,$
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DETERMINISM UNCERTAINT

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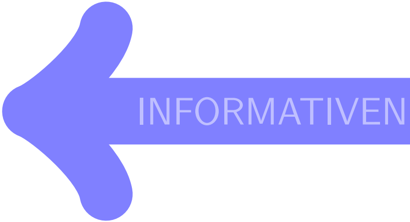

INFORMATIVEN

Three different levels of knowledge

DETERMINISM UNCERTAINT

EXPRESSIVENE

Three different levels of knowledge

DETERMINISM

UNCERTAINTY

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DETERMINISM

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limit of
of avail
(e.g., ver

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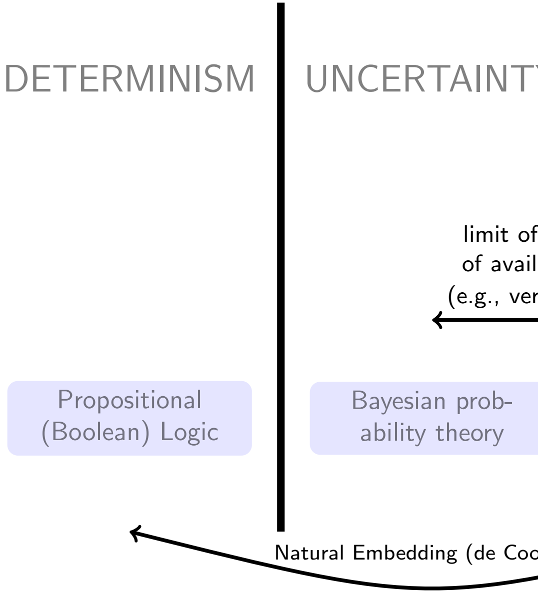
Propositional (Boolean) Logic

Bayesian probability theory

Three different levels of knowledge



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Natural Embedding (de Coo

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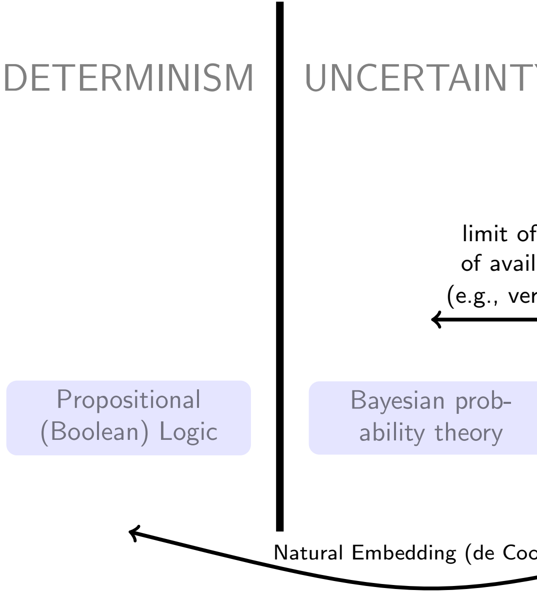

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Propositional (Boolean) Logic

Bayesian probability theory

Natural Embedding (de Coo

[. . .] Bayesian inference will always be a basic tool for practical everyday statistics,

if only because questions must be answered and decisions must be taken, so that a statistician must always stand ready to upgrade his vaguer forms of belief into precisely additive probabilities

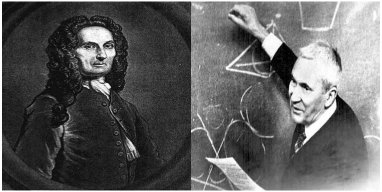
Art Dempster (in his foreword to Shafer's book)

Probability: one word for two (not exclusive) things

Randomness

Variability captured through repeated observations

De Moivre and Kolmogorov



- Chances
- Feature of the world
- Aleatory or objective
- Frequentist theory
- Limiting frequencies

Partial knowledge

Incomplete information about issues of interest

Bayes and De Finetti

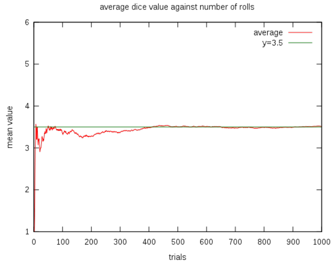
- Beliefs
- Feature of the observer
- Epistemic or subjective
- Bayesian theory
- Behaviour (bets dispositions)

Objective probability

- X taking its values in (finite set) Ω
- Value $X = x \in \Omega$ as the output of an experiment which can be iterated
- Prob $P(x)$ as limiting frequency

$$P(x) := \lim_{N \to +\infty} \frac{\#(X = x)}{N}$$

- Kolmogorov's axioms follow from this
- Probability as a property of the world
- Not only (statistical and quantum) mechanics, hazard games (coins, dices, cards), but also economics, bio/psycho/sociology, linguistics, etc.
- But not all events can be iterated ...



1. $\forall A \in 2^{\Omega},\ 0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $\forall A, B \in 2^{\Omega} : A \wedge B = \emptyset$
   $P(A \vee B) = P(A) + P(B)$

Probability in everyday life

Bersani: Il voto a primavera? "Ci sono buone probabilità"

Il segretario del Pd rilancia il progetto del Nuovo Ulivo e offre una proposta di alleanza di governo rivolta al cosiddetto Terzo Polo. Poi sdrammatizza con una battuta le voci sul futuro leader del partito e rilancia le primarie: "Una cosa bellissima"

10 ottobre, 2010

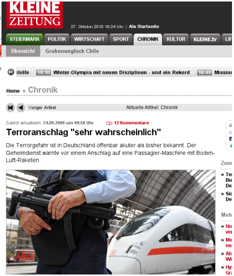
Il segretario del Pd Pier Luigi Bersani

Chile: probabilidades de encontrar mineros con vida 'son bajas'

Publicado el 12/Agosto/2010 | 09:10

SANTIAGO. **Las probabilidades de encontrar con vida a los 33 mineros atrapados desde hace una semana en un mina en el norte de Chile "son bajas"**, admitió el ministro de Minería, Laurance Golborne, que encabeza las labores de búsqueda.

"Las probabilidades de encontrarlos con vida son bajas", señaló el ministro en una entrevista con Canal 13, siete días después que un derrumbe sepultó a 33 mineros en el yacimiento San José, en las cercanías de Copiapó, a unos 800 km al norte de Santiago.



Zuletzt aktualisiert: **24.09.2009 um 09:58 Uhr**

Terroranschlag "sehr wahrscheinlich"

Die Terrorgefahr ist in Deutschland offenbar akuter als bisher bekannt. Der Geheimdienst warnte vor einem Anschlag auf eine Passagier-Maschine mit Boden-Luft-Raketen.

Xi Jinping, très probable successeur de Hu Jintao et "fils de prince"

AFP | 18.10.10 | 10h36

Xi Jinping, 57 ans, qui vient d'être nommé vice-président de la puissante Commission militaire

SPORTS

Wednesday October 20, 2010

Anderson and Sullivan both will 'probably' play for Herd

by Chuck McGill
Daily Mail sports writer

HUNTINGTON, W.Va. -- Regarding the quarterback

Economia

02/02/2010 às 16h05 - Atualizada em 25/10/2011 às 00h09

Fecombustíveis alerta para provável aumento de gasolina

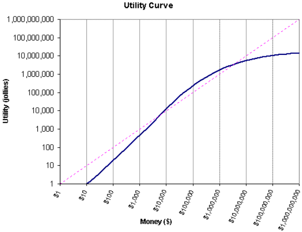
Probabilities often pertains to singular events

not necessarily related to statistics

Subjective probability

- Probability p of me smoking
- Singular event: frequency unavailable
- Subjective probability
  - models (partial) knowledge of a subject
  - feature of the subject not of the world
  - two subjects can assess different probs
- Quantitative measure of knowledge?
  - Behavioural approach
  - Subjective betting dispositions
  - A (linear) utility function is needed

- Money?
- Big money not linear!
- Small, somehow yes



lottery tickets

$\propto$

winning chance

$\propto$

benefit

infinite number of tickets
makes utility real-valued

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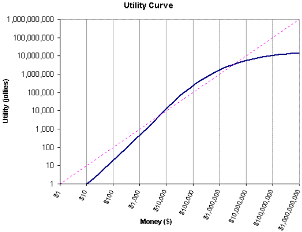

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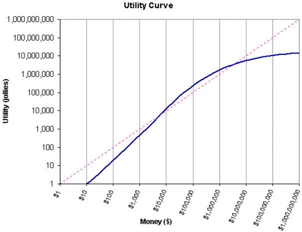

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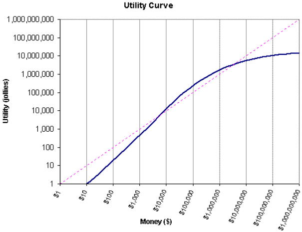

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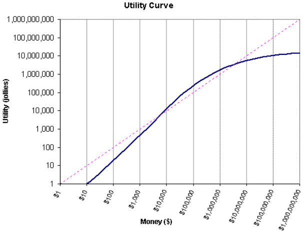

$$\begin{gathered}\text{lottery tickets}\\ \propto\\ \text{winning chance}\\ \propto\\ \text{benefit}\end{gathered}$$

infinite number of tickets makes utility real-valued

(Rationally) betting on gambles

- Probabilities as dispositions to buy/sell gambles
- Gambles as checks whose amount is uncertain/unknown

This check has a value of 100 EUR
if Alessandro is a smoker
zero otherwise

- The bookie sells this gamble
- Probability *p* as a *price* for the gamble
  - $\frac{\text{maximum price}}{\text{100EUR}}$ for which you buy the gamble
  - $\frac{\text{minimum price}}{\text{100EUR}}$ for which you (bookie) sell it
- Interpretation + rationality produce axioms

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  - $\frac{\text{maximum price}}{100\text{EUR}}$ for which you buy the gamble
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- Interpretation + rationality produce axioms

(Rationally) betting on gambles

- Probabilities as dispositions to buy/sell gambles
- Gambles as checks whose amount is uncertain/unknown

This check has a value of 100 EUR
if Alessandro is a smoker
zero otherwise

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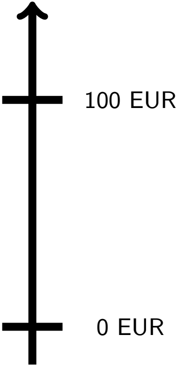

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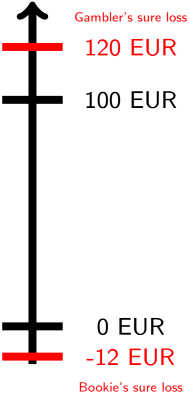

Gambler's sure loss

120 EUR

100 EUR

0 EUR

-12 EUR

Bookie's sure loss

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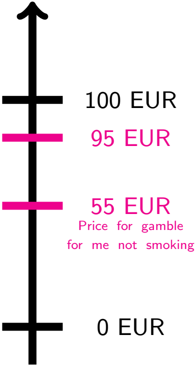

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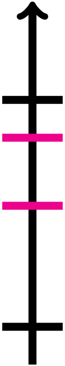

100 EUR

95 EUR

55 EUR

Price for gamble for me not smoking

0 EUR

You spend 150 EUR to (certainly) win 100 EUR!

Coherence and linear previsions

Don't be crazy: choose prices s.t.
there is always a chance to win
(whatever the stakes set by the bookie)

Prices $\{P_{A_i}\}_{i=1}^N$ for events $A_i \subseteq \Omega$, $i = 1, \ldots, N$ are coherent iff

$$\max_{x\in\Omega} \sum_{i=1}^{N} c_i [I_{A_i}(x) - P_{A_i}] \geq 0$$

Moreover, assessments $\{P_{A_i}\}_{i=1}^N$ are coherent iff

- Exists probability mass function $P(X)$: $P(A_i) = P_{A_i}$
- Or, for general gambles, linear functional $P(f_i) := P_{f_i}$

$$P(f) = \sum_{x\in\Omega} P(x)\, f(x)$$

linear prevision
to be extended to a
coherent lower prevision

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(subjective, behavioural) imprecise probabilities

De Finetti's precision dogma

$\overline{P}(x)$		$\underline{P}(x)$
minimum selling price	$=$	maximum buying price

Walley's proposal for imprecision

No strong reasons for that rationality only requires
$\underline{P}(x) \leq \overline{P}(x)$

- Avoid sure loss! With max buying prices $\underline{P}(A)$ and $\underline{P}(A^c)$, you can buy both gambles and earn one for sure:

$$\underline{P}(A) + \underline{P}(A^c) \leq 1$$

- Be coherent! When buying both A and B, you pay $\underline{P}(A) + \underline{P}(B)$ and you have a gamble which gives one if $A \cup B$ occurs:

$$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$$

coherence self-consistency (beliefs revised if unsatisfied)
less problematic than a.s.l.

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(Some) Reasons for imprecise probabilities

- Reflect the amount of information on which probs are based
- Uniform probs model indifference not ignorance
- When doing introspection, sometimes indecision/indeterminacy
- Easier to assess (e.g., qualitative knowledge, combining beliefs) Assessing precise probs could be possible in principle, but not in practice because of our bounded rationality
- Natural extension of precise models defined on some events determine only imprecise probabilities for events outside
- Robustness in statistics (multiple priors/likelihoods) and decision problems (multiple prob distributions/utilities)

Credal sets (Levi, 1980) as IP models

- Without the precision dogma, incomplete knowledge described by (credal) sets of probability mass functions
- Induced by a finite number of assessments (l/u gambles prices) which are linear constraints on the consistent probabilities
- Sets of consistent (precise) probability mass functions convex with a finite number of extremes (if $|\Omega| < +\infty$)
- E.g., no constraints ⇒ vacuous credal set (model of ignorance)

$$K(X) = \left\{ P(X) \left| \begin{array}{l} \sum_{x\in\Omega} P(x) = 1 \\ P(x) \geq 0 \end{array} \right. \right\}$$

Natural extension

- Price assessments are linear constraints on probabilities (e.g., $\underline{P}(f) = .21$ means $\sum_x P(x)f(x) \geq .21$)
- Compute the extremes $\{P_j(X)\}_{j=1}^{v}$ of the feasible region
- The credal set $K(X)$ is $\mathrm{ConvHull}\{P_j(X)\}_{j=1}^{v}$
- Lower prices/expectations of any gamble/function of/on X

$$\underline{P}(h) = \min_{P(X) \in K(X)} \sum_{x \in \mathcal{X}} P(x) \cdot h(x)$$

LP task: optimum on the extremes of $K(X)$

Computing expectations (inference) on credal sets

- Constrained optimization problem, or
- Combinatorial optimization on the extremes space (# of extremes can be exponential in # of constraints)

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Lower-upper conjugacy

E.g., with events

$$\underline{P}(A) = \min_{P(X)\in K(X)} \sum_{x\in A} P(x)$$

$$\overline{P}(A^c) = \max_{P(X)\in K(X)} \sum_{x\notin A} P(x) = \max_{P(X)\in K(X)} \left[1 - \sum_{x\in A} P(x)\right] = 1 - \underline{P}(A)$$

For gambles, similarly,

$$\overline{P}(-f) = -\underline{P}(f)$$

$$\underline{P}(f) = \min_{P(X)\in K(X)} \sum_{x} P(x)f(x)$$

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Self-conjugacy $\equiv$ single-point (precise) credal set $\equiv$ linear functional

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$$\overline{P}(-f) = -\underline{P}(f)$$

$$\underline{P}(f) = \min_{P(X) \in K(X)} \sum_{x} P(x) f(x)$$

$$\overline{P}(-f) = \max_{P(X) \in K(X)} \sum_{x} [-P(x) f(x)] = - \min_{P(X) \in K(X)} \sum_{x} P(x) f(x)$$

Self-conjugacy $\equiv$ *single-point (precise) credal set* $\equiv$ *linear functional*

Credal Sets over Boolean Variables

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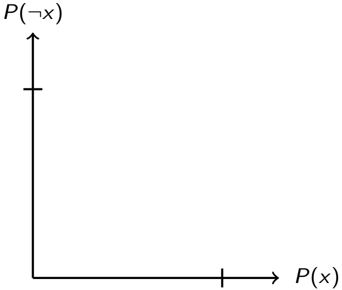

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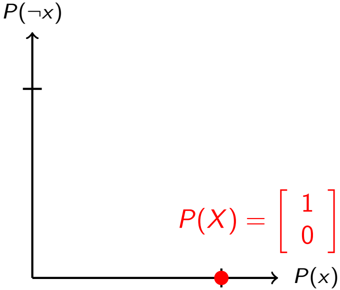

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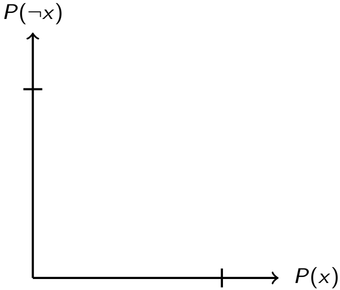

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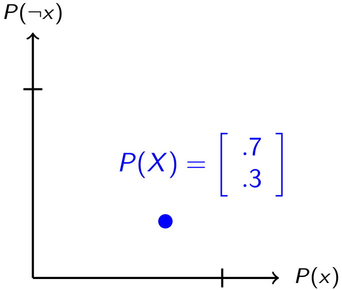

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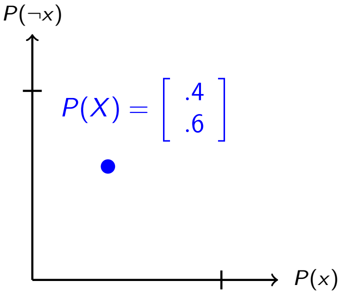

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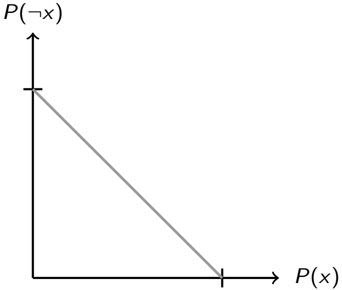

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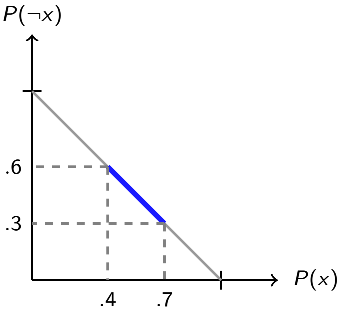
$$K(X) \equiv \left\{ P(X) = \left[\begin{array}{c} p \\ 1-p \end{array} \right] \middle| .4 \leq p \leq .7 \right\}$$



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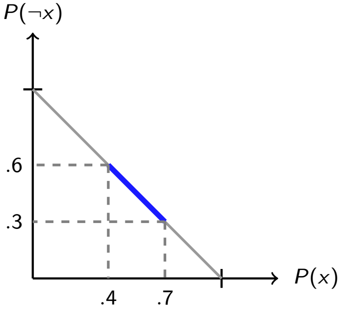
  $$K(X) \equiv \left\{ P(X) = \left[\begin{array}{c} p \\ 1-p \end{array} \right] \middle| .4 \leq p \leq .7 \right\}$$
- A CS over a Boolean variable cannot
  have more than two vertices!

  $$\mathrm{ext}[K(X)] = \left\{ \left[\begin{array}{c} .7 \\ .3 \end{array} \right], \left[\begin{array}{c} .4 \\ .6 \end{array} \right] \right\}$$



Geometric Representation of CSs (ternary variables)

- Ternary X (e.g., $\Omega = \{$win,draw,loss$\}$)
- $P(X) \equiv$ point in the space (simplex)
- No bounds to $|\mathrm{ext}[K(X)]|$
- Modeling ignorance
  - Uniform models indifference
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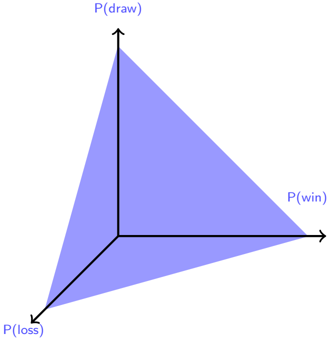

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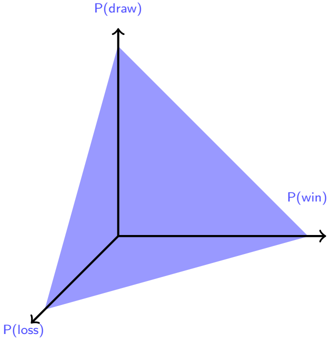

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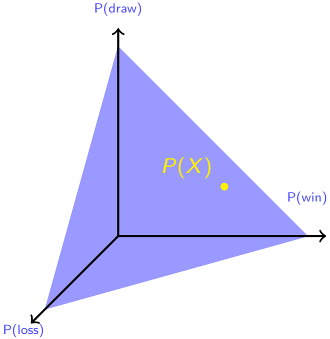

$$P(X) = \left[\begin{array}{c} .6 \\ .3 \\ .1 \end{array}\right]$$

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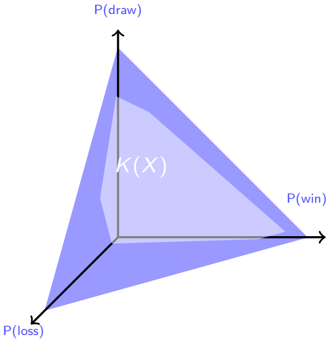

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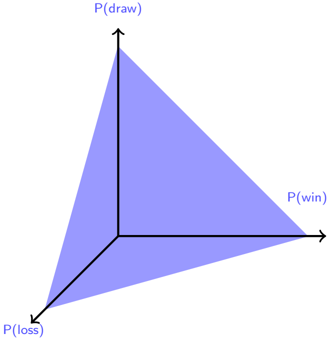

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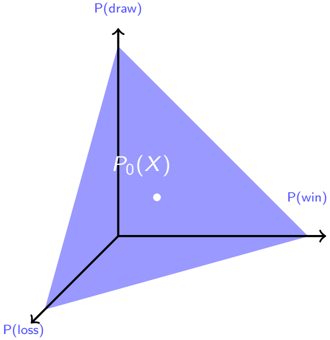

$$P_0(x) = \frac{1}{|\Omega_X|}$$

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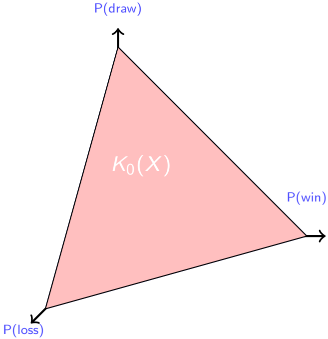

$$K_0(X)=\left\{P(X) \;\middle|\; \begin{array}{l} \sum_x P(x) = 1, \\ P(x) \geq 0 \end{array} \right\}$$

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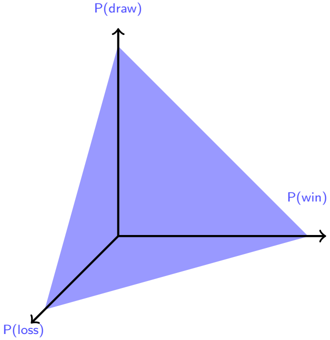

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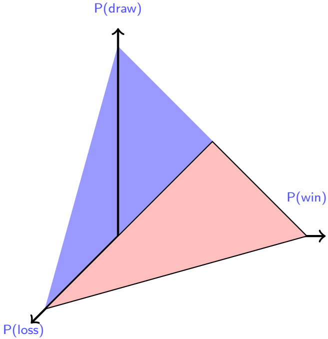

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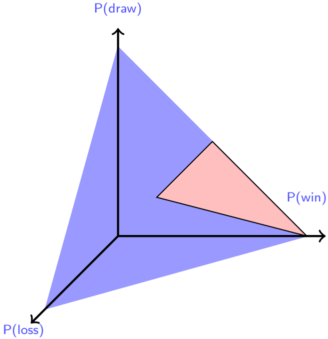

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From natural language to linear constraints on probabilities

(Walley, 1991)

extremely probable $P(x) \geq 0.98$
very high probability $P(x) \geq 0.9$
highly probable $P(x) \geq 0.85$
very probable $P(x) \geq 0.75$
has a very good chance $P(x) \geq 0.65$
quite probable $P(x) \geq 0.6$
probable $P(x) \geq 0.5$
has a good chance $0.4 \leq P(x) \leq 0.85$
is improbable (unlikely) $P(x) \leq 0.5$
is somewhat unlikely $P(x) \leq 0.4$
is very unlikely $P(x) \leq 0.25$
has little chance $P(x) \leq 0.2$
is highly improbable $P(x) \leq 0.15$
is has very low probability $P(x) \leq 0.1$
is extremely unlikely $P(x) \leq 0.02$

Marginalization (and credal sets in 4D)

- Two Boolean variables: **S**moker, Lung **C**ancer
- 8 “Bayesian” physicians, each assessing $P_j(S, C)$

$$K(S, C) = \mathrm{CH}\left\{P_j(S, C)\right\}_{j=1}^{8}$$

j	$P_j(s, c)$	$P_j(s, \bar{c})$	$P_j(\bar{s}, c)$	$P_j(\bar{s}, \bar{c})$
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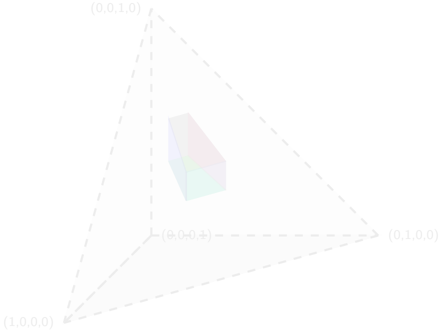
Marginals elementwise (on extremes)

$$K(C) = CH\left\{\sum_s P_j(C, s)\right\}_{j=1}^{8} \qquad \frac{1}{2} \leq P(c) \leq \frac{3}{4}$$



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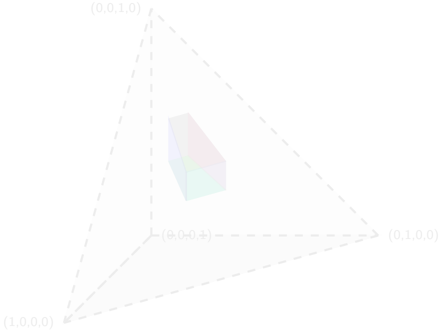

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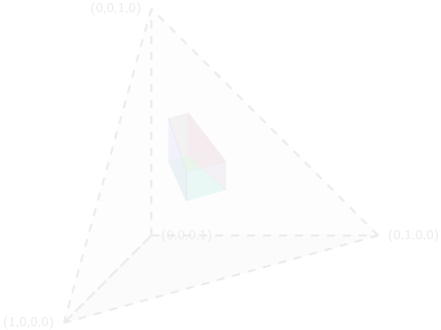

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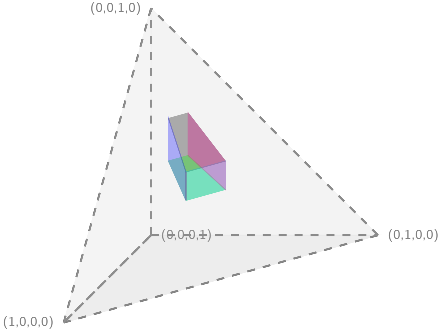

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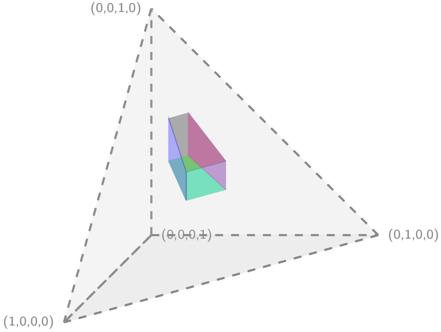

Credal sets induced by probability intervals

- Assessing lower and upper probabilities: $[l_x, u_x]$, for each $x \in \Omega$
- The consistent credal set is

$$K(X) := \left\{ P(X) \,\middle|\, \begin{array}{l} l_x \leq P(x) \leq u_x \\ P(x) \geq 0 \\ \sum_x P(x) = 1 \end{array} \right\}$$

- Avoiding sure loss implies non-emptiness of the credal set

$$\sum_x l_x \leq 1 \leq \sum_x u_x$$

- Coherence implies the reachability (bounds are tight)

$$u_x + \sum_{x' \neq x} l_x \leq 1 \qquad l_x + \sum_{x' \neq x} u_x \geq 1$$

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Credal sets induced by probability intervals

- Assessing lower and upper probabilities: $[l_x, u_x]$, for each $x \in \Omega$
- The consistent credal set is

$$K(X) := \left\{ P(X) \;\middle|\; \begin{array}{l} l_x \leq P(x) \leq u_x \\ P(x) \geq 0 \\ \sum_x P(x) = 1 \end{array} \right\}$$

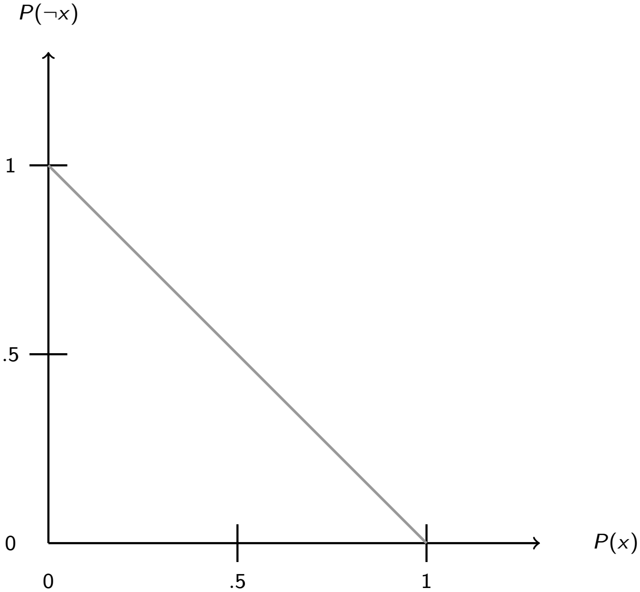
- Avoiding sure loss implies non-emptiness of the credal set

$$\sum_x l_x \leq 1 \leq \sum_x u_x$$

- Coherence implies the reachability (bounds are tight)

$$u_x + \sum_{x' \neq x} l_x \leq 1 \qquad l_x + \sum_{x' \neq x} u_x \geq 1$$

Refining assessments (when possible)



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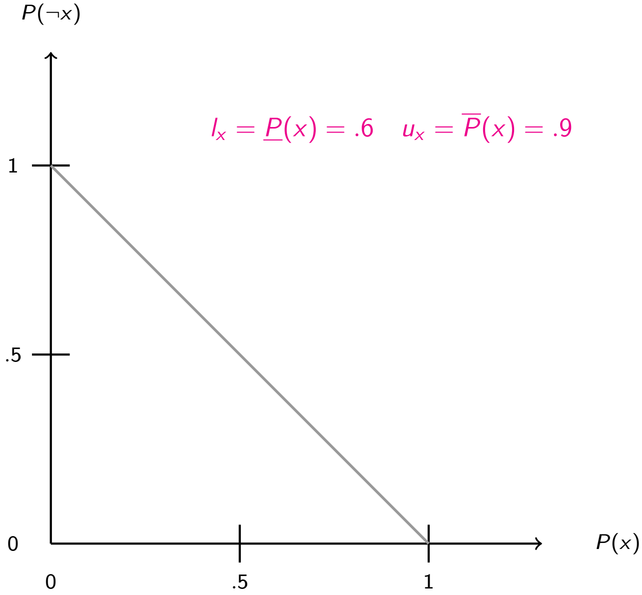

$l_x = \underline{P}(x) = .6 \quad u_x = \overline{P}(x) = .9$

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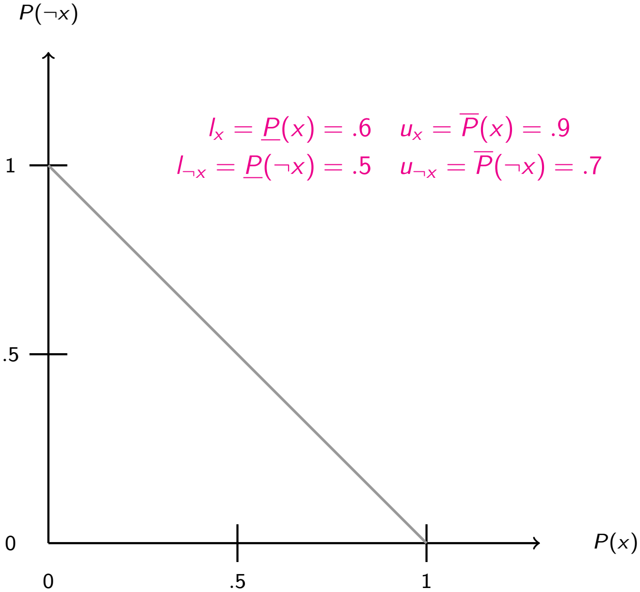

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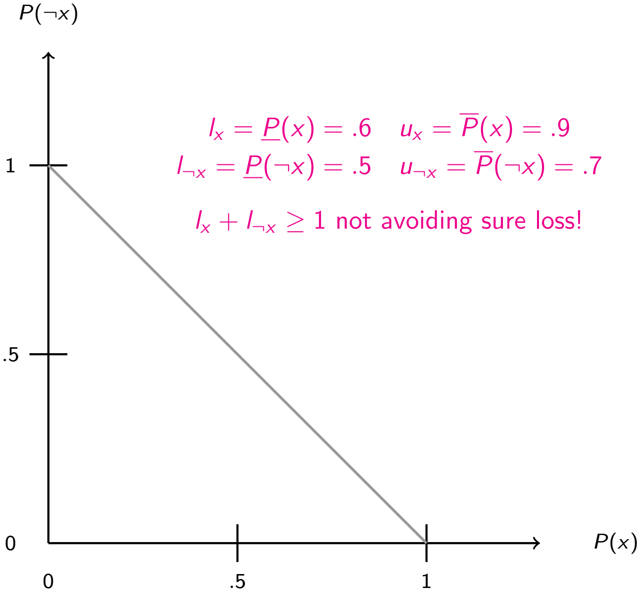

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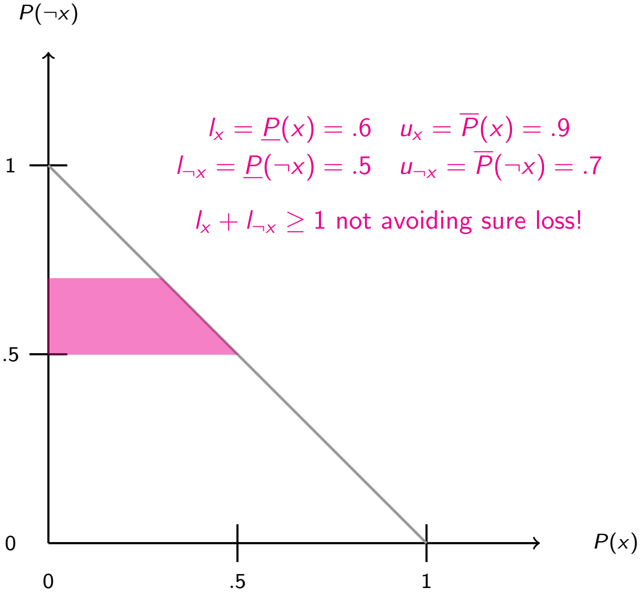

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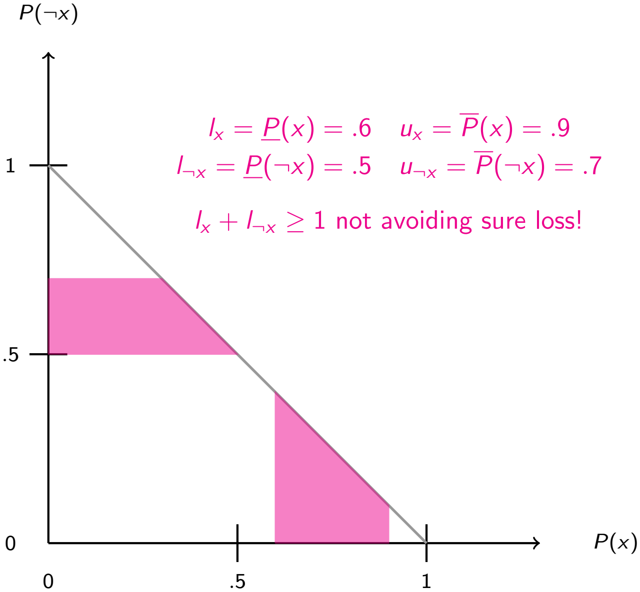

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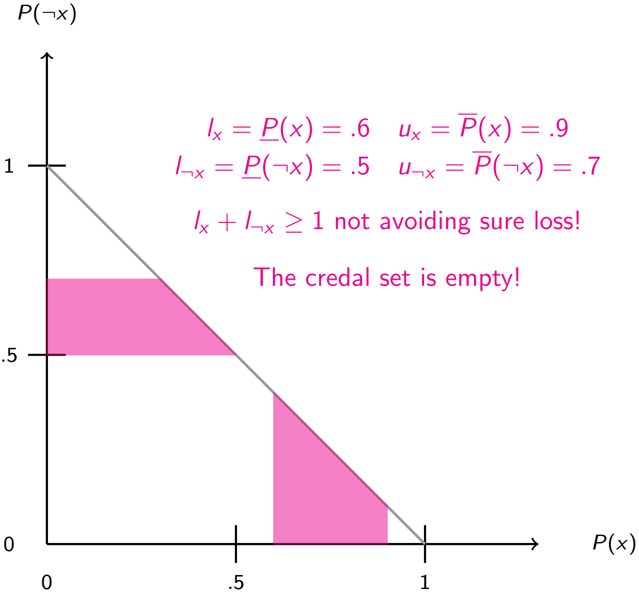

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The credal set is empty!

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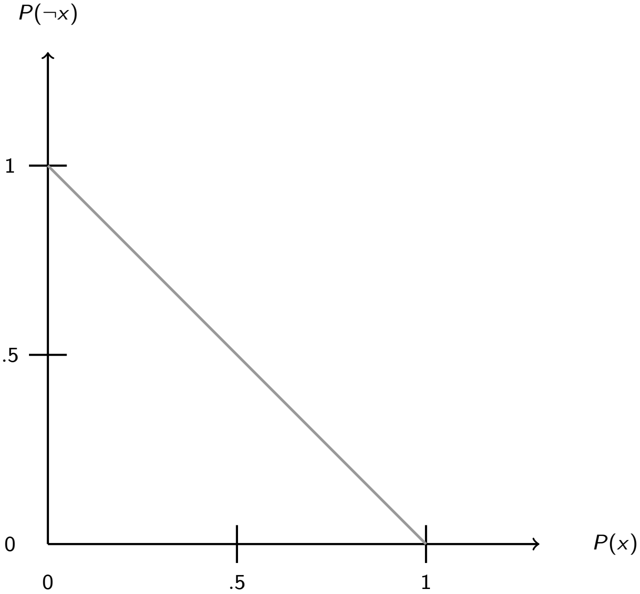

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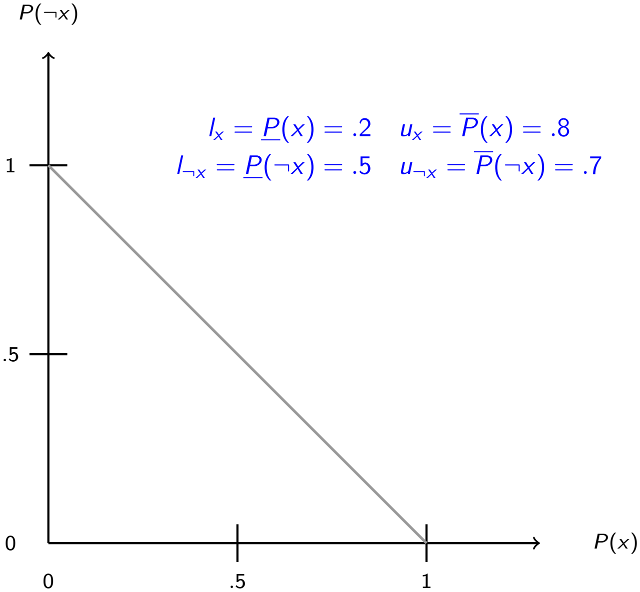

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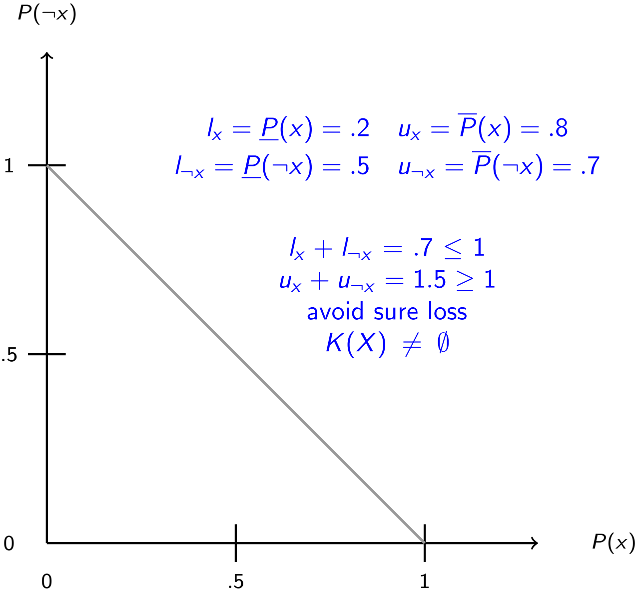

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avoid sure loss

$$K(X) \neq \emptyset$$

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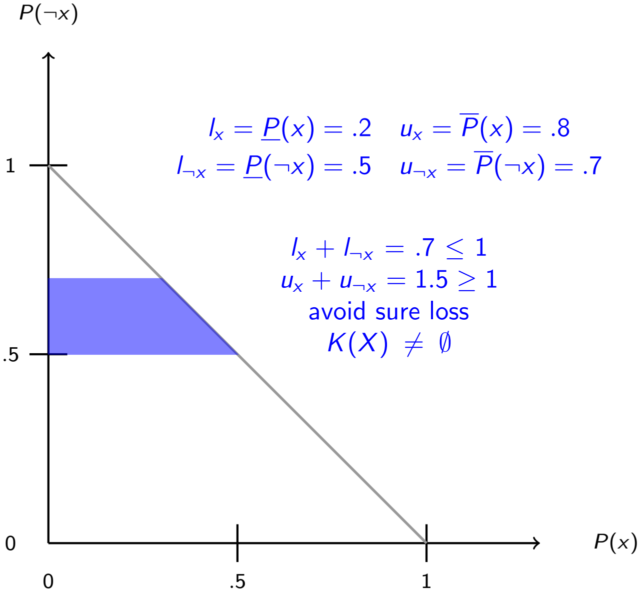

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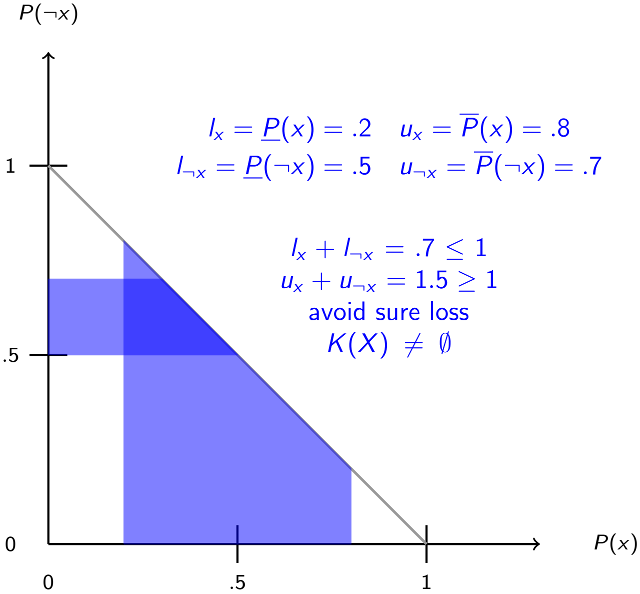

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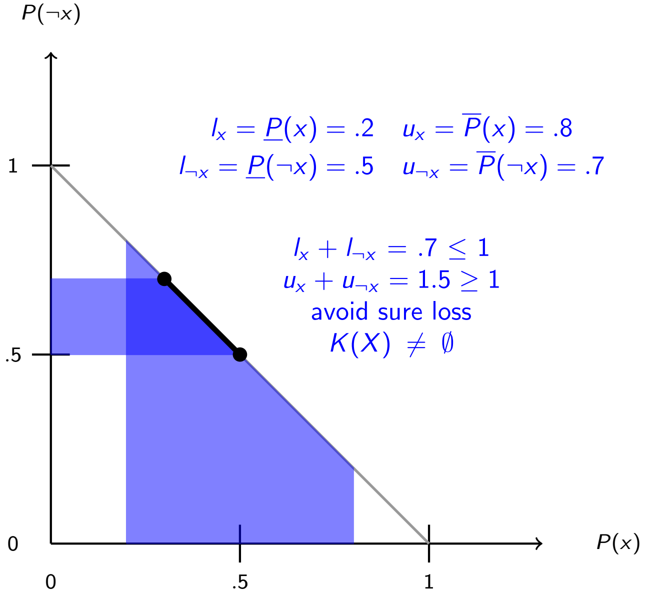

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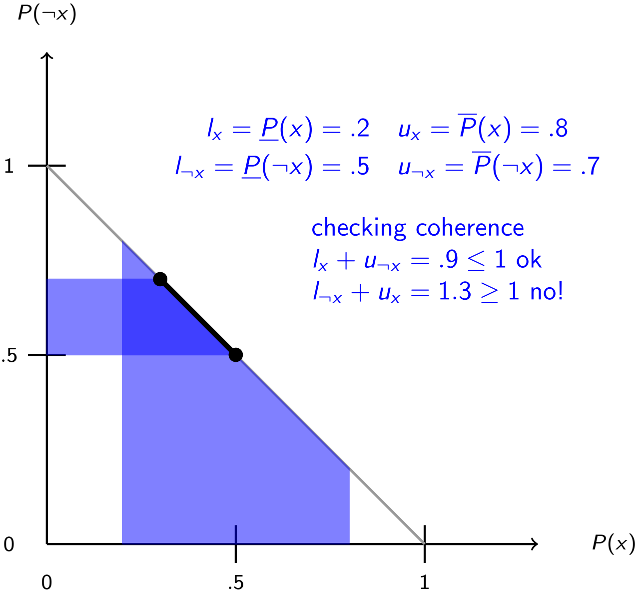

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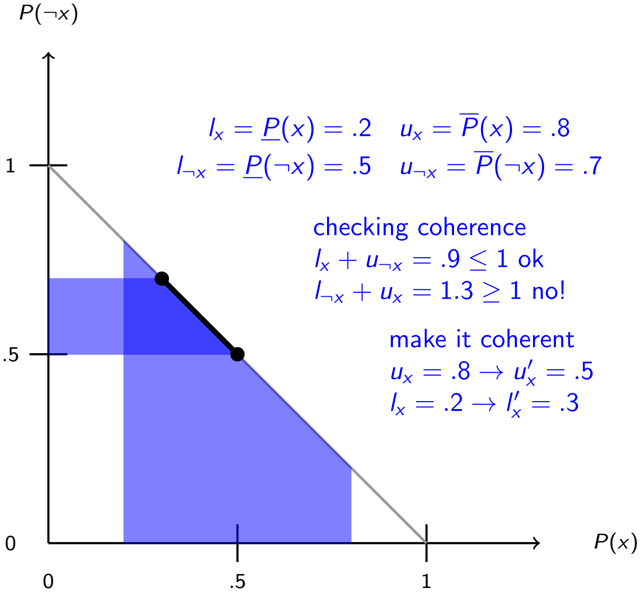

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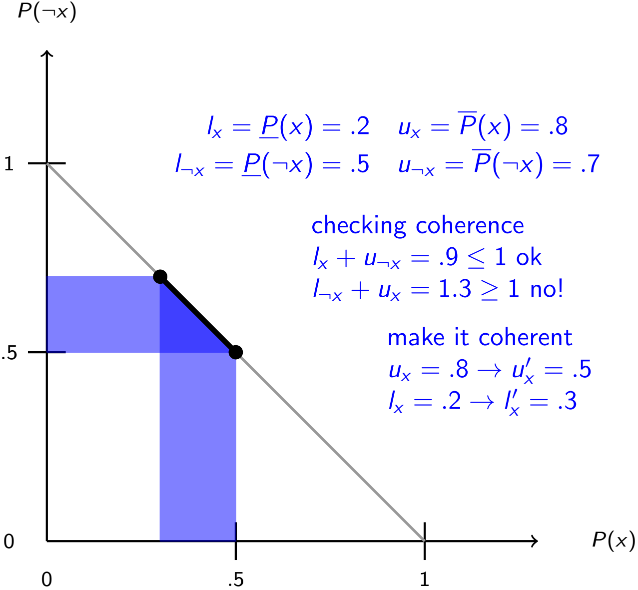

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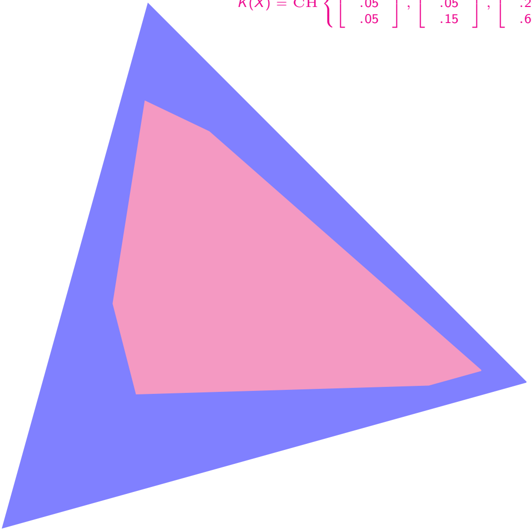
Probability intervals are not fully general

$$K(X) = \mathrm{CH}\left\{ \begin{bmatrix} .90 \\ .05 \\ .05 \end{bmatrix}, \begin{bmatrix} .80 \\ .05 \\ .15 \end{bmatrix}, \begin{bmatrix} .20 \\ .20 \\ .60 \end{bmatrix}, \begin{bmatrix} .10 \\ .40 \\ .50 \end{bmatrix}, \begin{bmatrix} .05 \\ .80 \\ .15 \end{bmatrix}, \begin{bmatrix} .20 \\ .70 \\ .10 \end{bmatrix} \right\}$$



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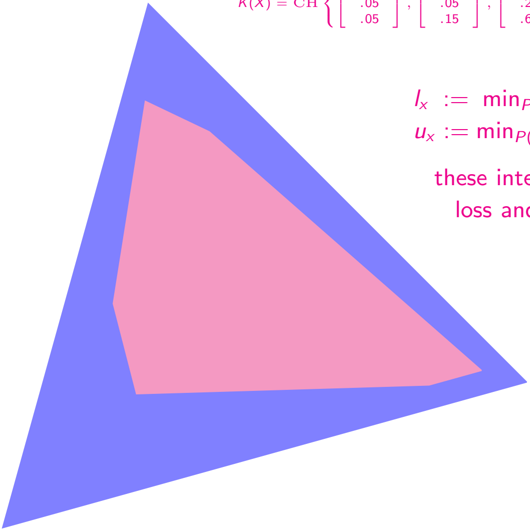

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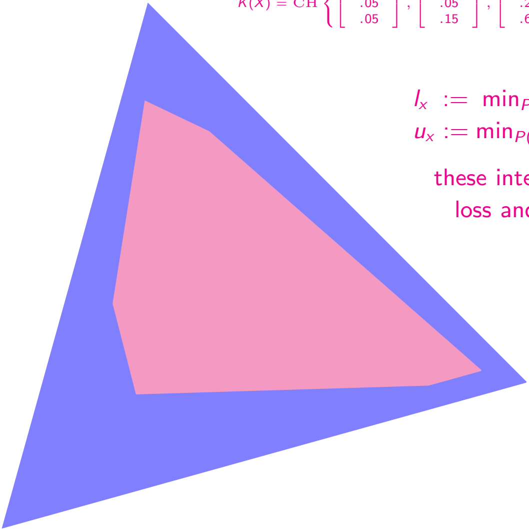

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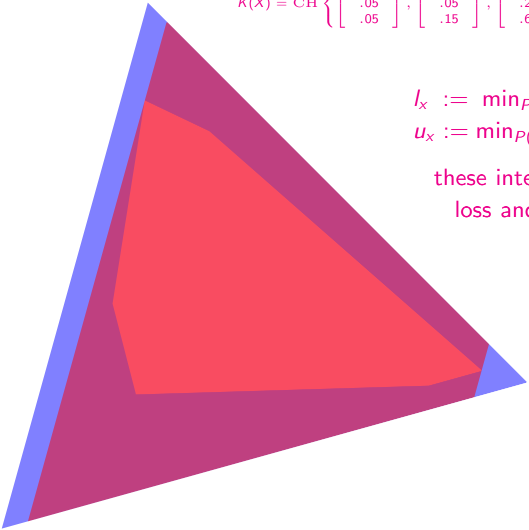

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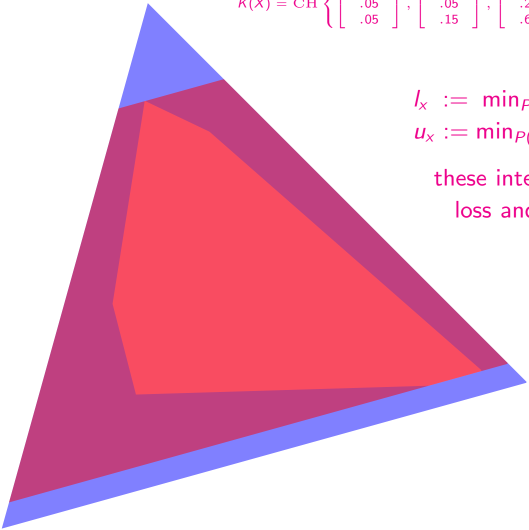

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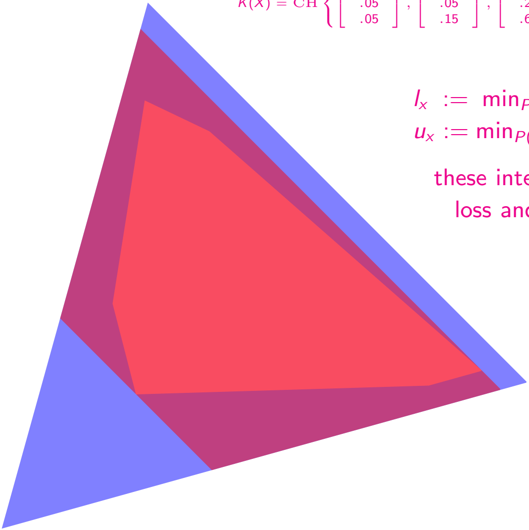

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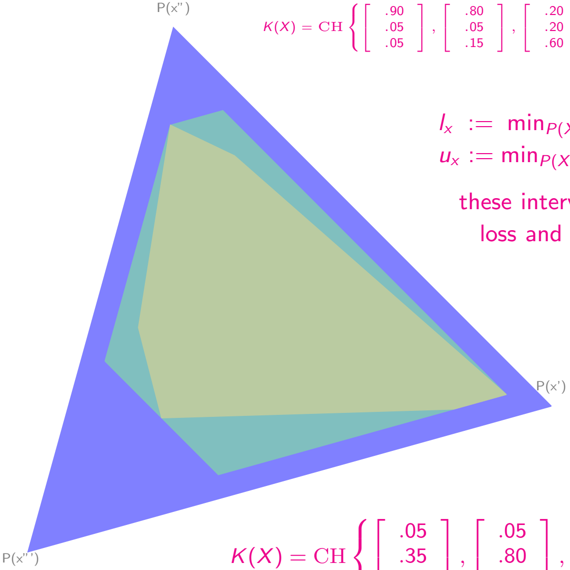

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Learning credal sets from (few) data

- Learning from data about X
- Max lik estimate $P(x) = \frac{n(x)}{N}$
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- Imprecise: set of priors (vacuous t)

$$\frac{n(x)}{N+s} \leq P(x) \leq \frac{n(x)+s}{N+s}$$

imprecise Dirichlet model
(Walley & Bernard)

- They a.s.l. and are coherent
- Non-negligible size of intervals only for small N
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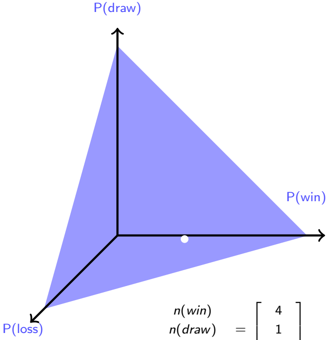

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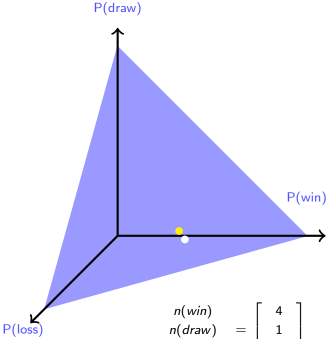

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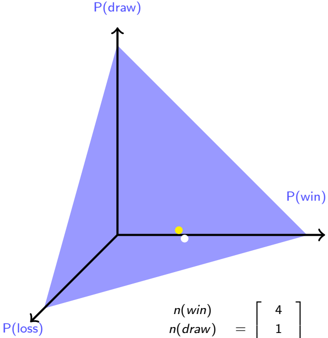

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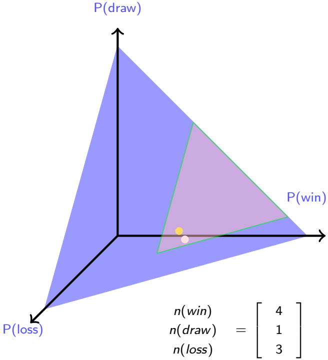

$$\begin{array}{c} n(win) \\ n(draw) \\ n(loss) \end{array} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

1957:	Spain	vs.	Italy	5 — 1
1973:	Italy	vs.	Spain	3 — 2
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1983:	Spain	vs.	Italy	1 — 0
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Learning credal sets from (few) data

- Learning from data about X
- Max lik estimate $P(x) = \frac{n(x)}{N}$
- Bayesian (ESS $s = 2$) $\frac{n(x)+st(x)}{N}$
- Imprecise: set of priors (vacuous t)

$$\frac{n(x)}{N+s} \leq P(x) \leq \frac{n(x)+s}{N+s}$$

imprecise Dirichlet model
(Walley & Bernard)

- They a.s.l. and are coherent
- Non-negligible size of intervals only for small N
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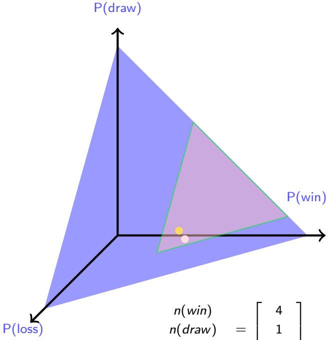

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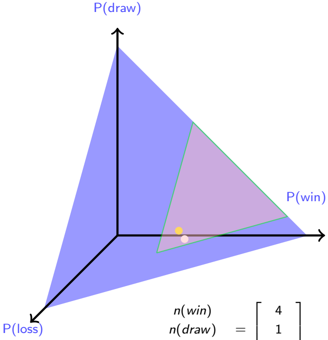

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Learning credal sets from (missing) data

- Coping with missing data?
- Missing at random (MAR) $P(O = *|X = x)$ indep of X Ignore missing data
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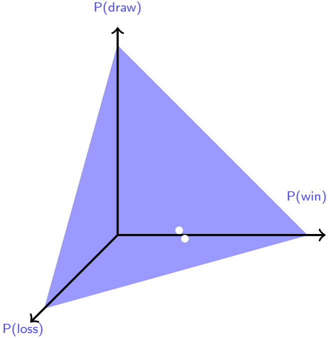

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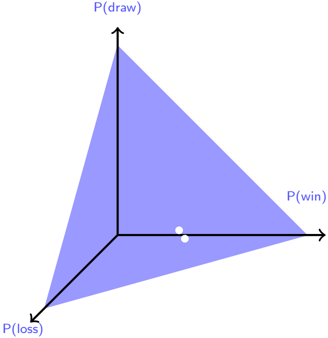

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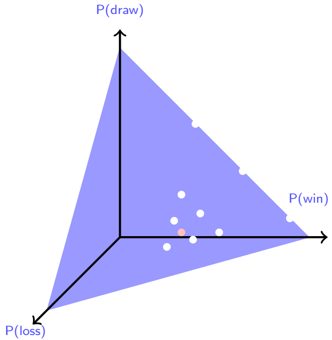

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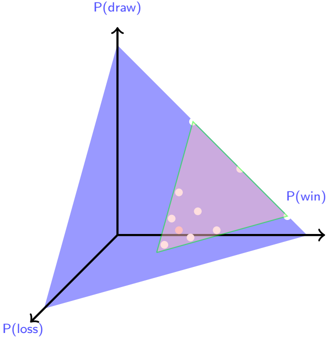

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Basic operations with credal sets

	PRECISE Mass functions	IMPRECISE Credal sets
Joint	$P(X,Y)$	$K(X,Y)$
Marginalization	$P(X)$ s.t. $p(x) = \sum_y p(x,y)$	$K(X) =$ $\left\{ P(X) \,\middle\vert\, \begin{array}{l} P(x) = \sum_y P(x,y) \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
Conditioning	$P(X\vert y)$ s.t. $p(x\vert y) = \frac{P(x,y)}{\sum_y P(x,y)}$	$K(X\vert y) =$ $\left\{ P(X\vert y) \,\middle\vert\, \begin{array}{l} P(x\vert y) = \frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
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Basic operations with credal sets (vertices)

IMPRECISE
Credal sets

IMPRECISE
Extremes

Joint

$$K(X,Y)$$

Marginalization

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Basic operations with credal sets (vertices)

	IMPRECISE Credal sets	IMPRECISE Extremes
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Marginalization	$K(X) = \left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y) \in K(X,Y) \end{array}\right\}$	$= \mathrm{CH}\left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y) \in \mathrm{ext}[K(X,Y)] \end{array}\right\}$
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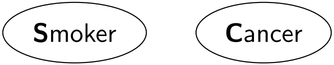
An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer



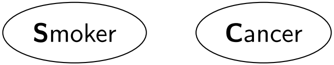
Smoker

Cancer

An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer

j	$P_j(s, c)$	$P_j(s, \neg c)$	$P_j(\neg s, c)$	$P_j(\neg s, \neg c)$
1	1/8	1/8	3/8	3/8
2	1/8	1/8	9/16	3/16
3	3/16	1/16	3/8	3/8
4	3/16	1/16	9/16	3/16
5	1/4	1/4	1/4	1/4
6	1/4	1/4	3/8	1/8
7	3/8	1/8	1/4	1/4
8	3/8	1/8	3/8	1/8



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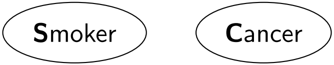
3	3/16	1/16	3/8	3/8
4	3/16	1/16	9/16	3/16
5	1/4	1/4	1/4	1/4
6	1/4	1/4	3/8	1/8
7	3/8	1/8	1/4	1/4
8	3/8	1/8	3/8	1/8



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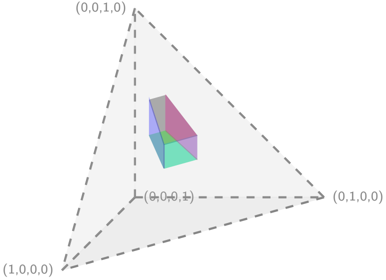
Cancer



An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer
- Compute:
  - Marginal $K(S)$
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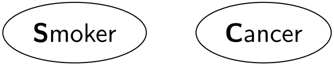
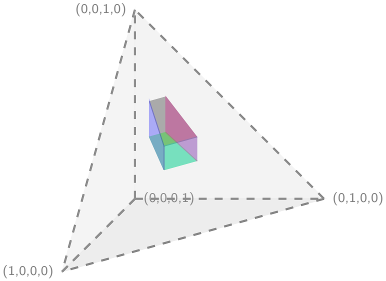


j	$P_j(s, c)$	$P_j(s, \neg c)$	$P_j(\neg s, c)$	$P_j(\neg s, \neg c)$
1	1/8	1/8	3/8	3/8
2	1/8	1/8	9/16	3/16
3	3/16	1/16	3/8	3/8
4	3/16	1/16	9/16	3/16
5	1/4	1/4	1/4	1/4
6	1/4	1/4	3/8	1/8
7	3/8	1/8	1/4	1/4
8	3/8	1/8	3/8	1/8



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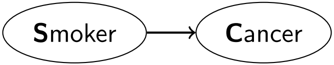
- Is this a (I)PGM?



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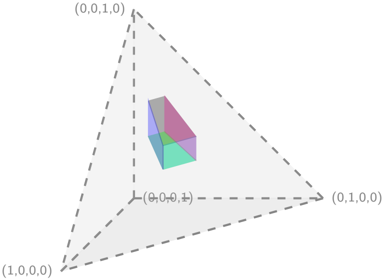

Cano-Cano-Moral Transformation

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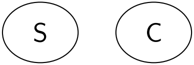

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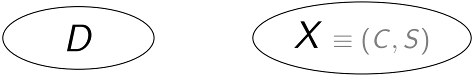

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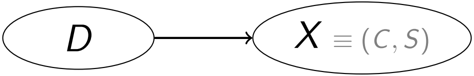

unconditional
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conditional
precise

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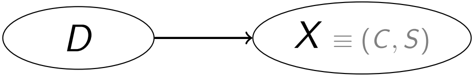

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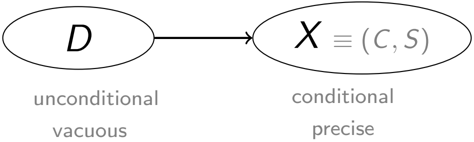

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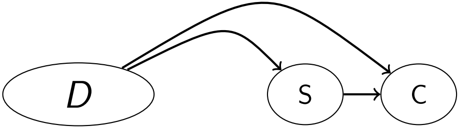

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aka Decomposable Multivariate Probabilistic Models

(whose decomposability is induced by independence)

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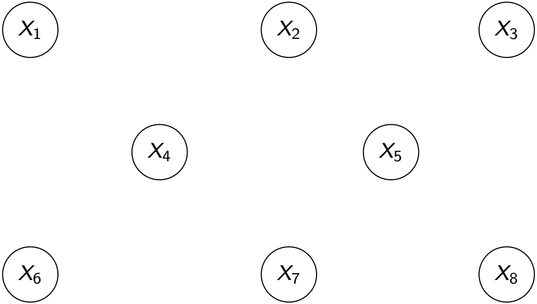

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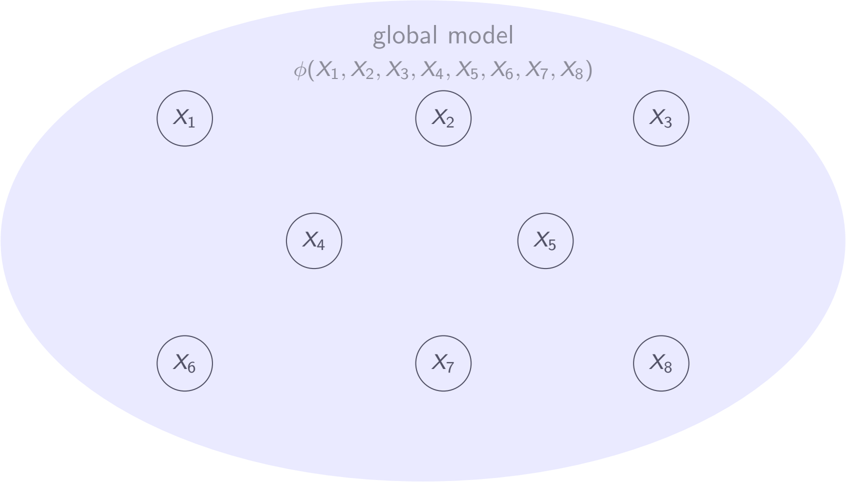

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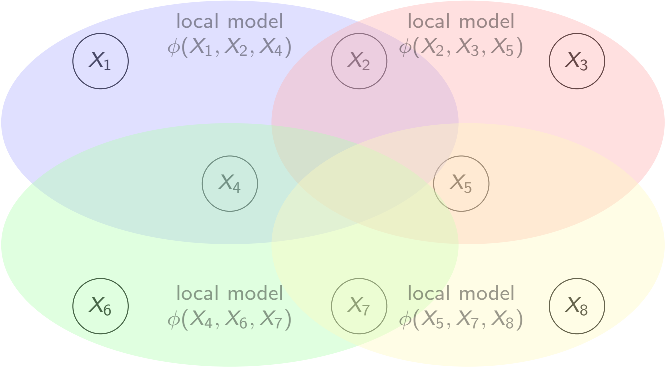
$$\phi(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = \phi(X_1, X_2, X_4) \otimes \phi(X_2, X_3, X_5) \otimes \phi(X_4, X_6, X_7) \otimes \phi(X_5, X_7, X_8)$$



Probabilistic Graphical Models

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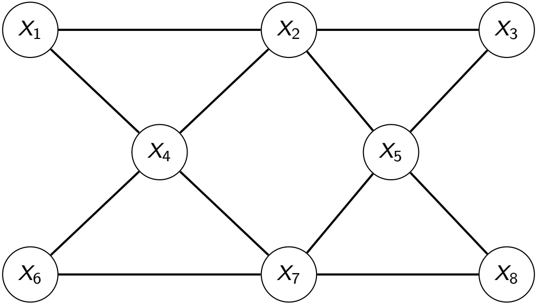
undirected graphs

precise/imprecise Markov random fields



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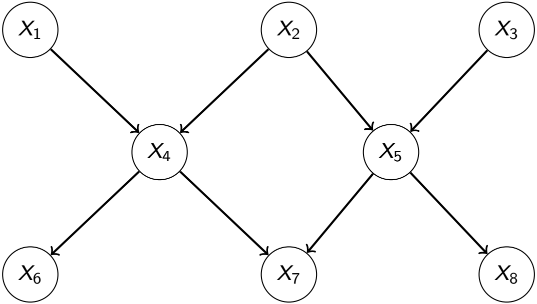
Bayesian/credal networks



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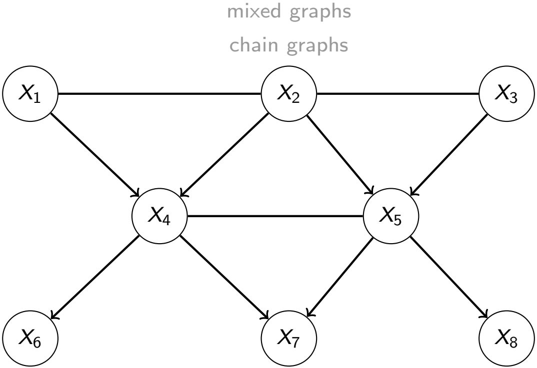

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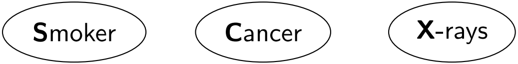
Epistemic irrelevance (imprecise case)

- Y epistemically irrelevant to X: $K(X|y) = K(X)$
- Asymmetric concept! Its simmetrization: epistemic indep

Every notion of independence/irrelevance admits a conditional

A tri-variate example

- 3 Boolean variables: **S**moker, Lung **C**ancer, **X**-rays



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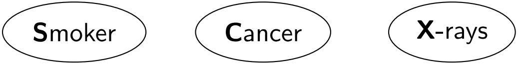

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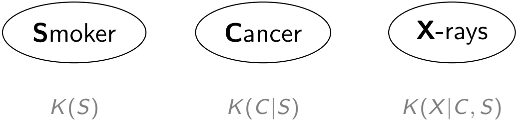

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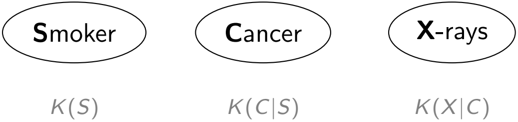

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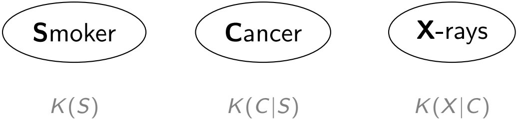

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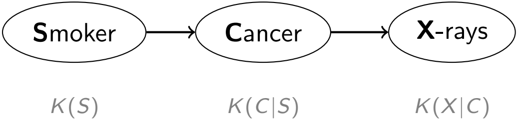
- A true PGM! Needed: language to express independencies



Markov Condition

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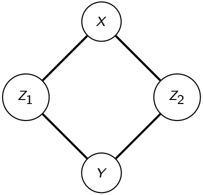

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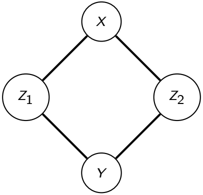

Directed Graphs

Given its parents, every node is independent of its
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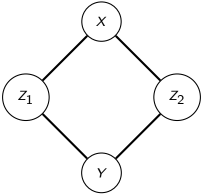

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X and Y are d-separated by Z if, along every path between X and Y there is a W such that either W has converging arrows and is not in Z and none of its descendants are in Z, or W has no converging arrows and is in Z

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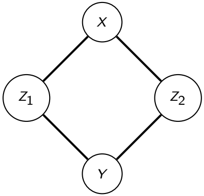

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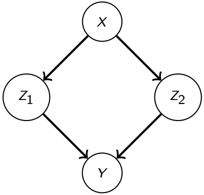

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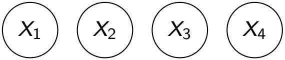

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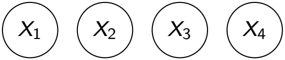

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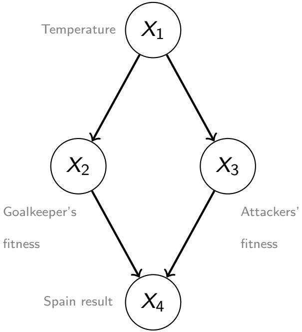

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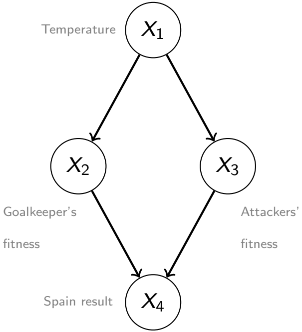

E.g., given temperature, fitnesses independent

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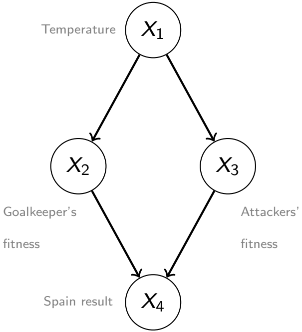

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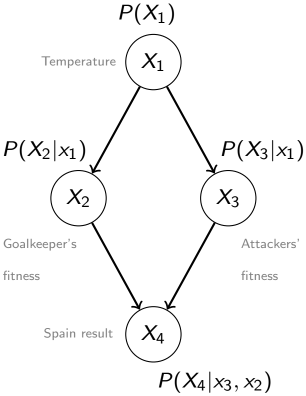

Bayesian networks *(Pearl, 1986)*

- Set of categorical variables $X_1, \ldots, X_n$
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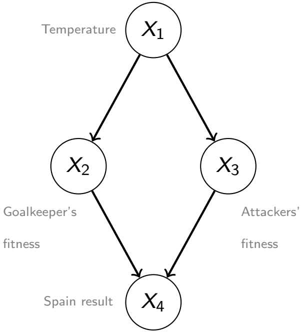

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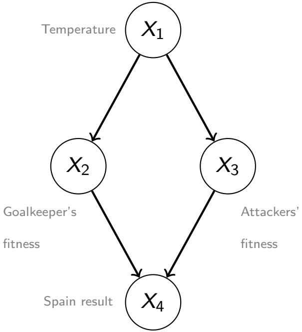

$$P(x_1, x_2, x_3, x_4) = P(x_1)P(x_2|x_1)P(x_3|x_1)P(x_4|x_3, x_2)$$

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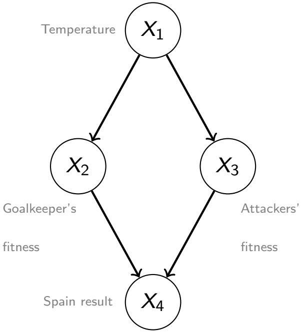

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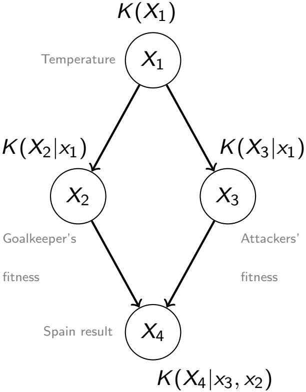

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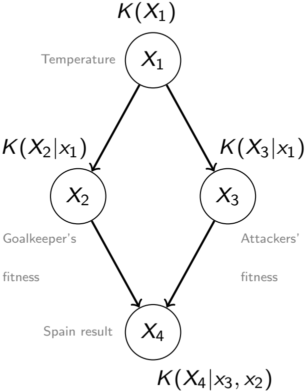

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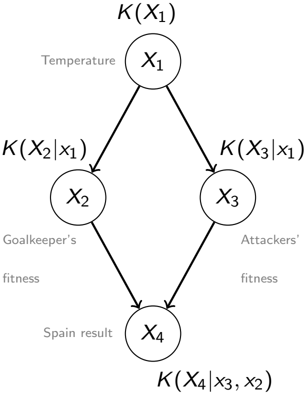

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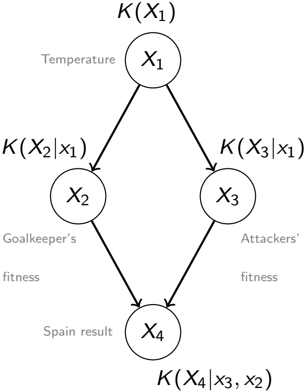
- Every conditional mass function takes values in its credal set independently of the others
  CN $\equiv$ (exponential) number of BNs



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Non-separately specified CNs

- Constraints among different conditional mass functions of a CN
- Explicit enumeration of the relative BNs
  - Auxiliary parent selecting the conditional probabilities *(Cano, Cano, Moral, 1994)* with a vacuous prior
- "Extensive" specification
  - Constraints among conditional mass functions of the same variable
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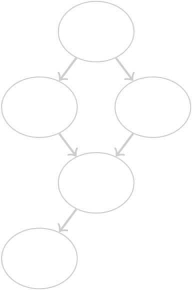

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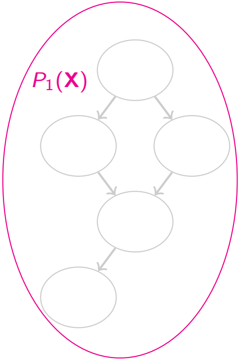

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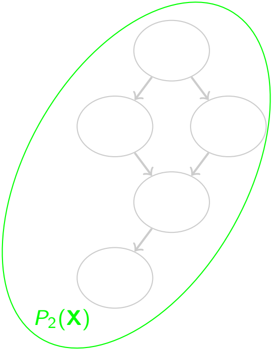

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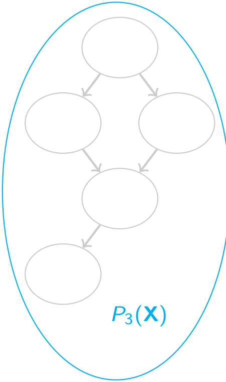

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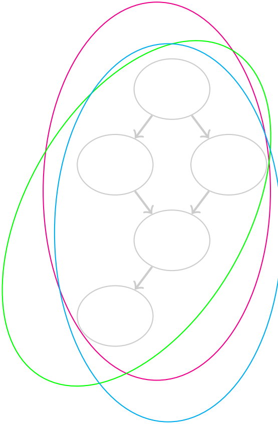

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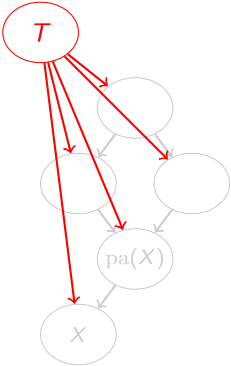

$$P(X|\mathrm{pa}(X),\ T = t_j\ ) = P_j(X|\mathrm{pa}(X))$$

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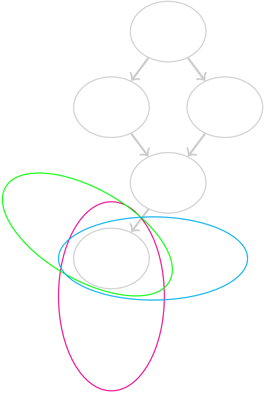

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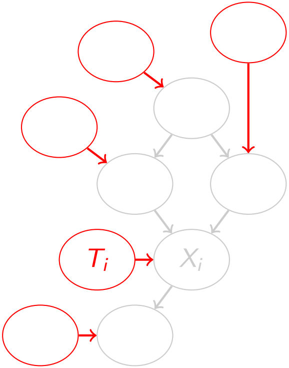

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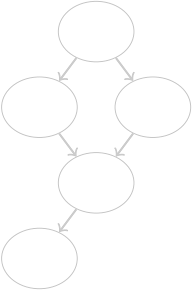
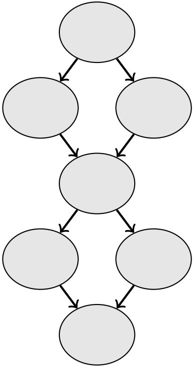


Updating credal networks



- Conditional probs for a variable of interest X_q given observations $X_E = x_E$
- Updating Bayesian nets is NP-hard
  (fast algorithms for polytrees)

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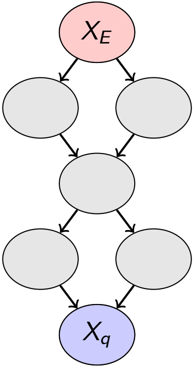

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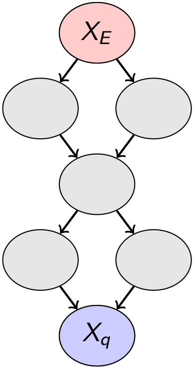

$P(x_q|x_E) = .38$

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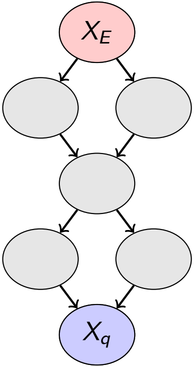

$.21 \leq P(X_q|x_E) \leq .46$

Medical diagnosis by CNs (a simple example of)

- Five Boolean vars
- Conditional independence relations by a DAG
- Elicitation of the local (conditional) CSs
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- The strong extension $K(S,C,B,X,D) =$

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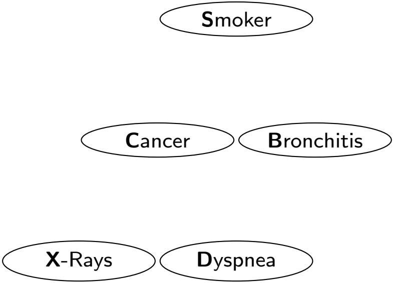

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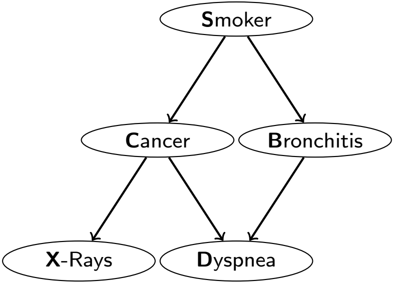

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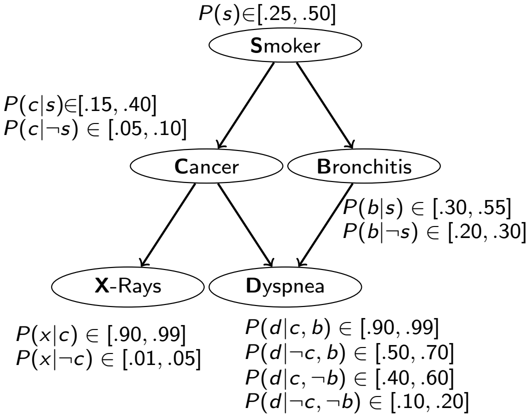

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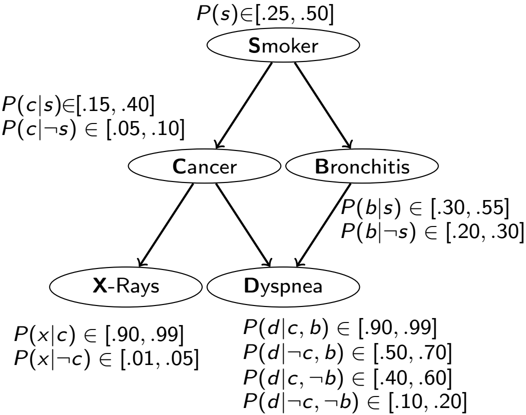

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X-Rays

Dyspnea

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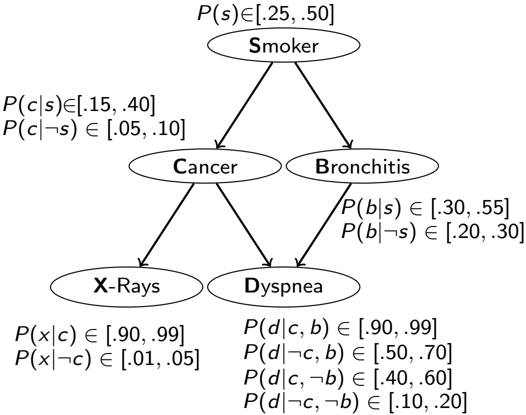
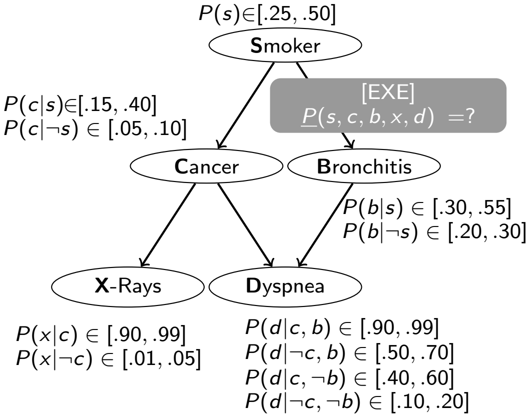


$$\mathrm{CH}\left\{P(S, C, B, X, D)\;\middle|\;\begin{array}{l} P(s, c, b, x, d)=P(s)P(c|s)P(b|s)P(x|c)P(d|c, b)\\ P(S) \in K(S)\\ P(C|s) \in K(C|s), P(C|\neg s) \in K(C|\neg s)\\ \ldots \end{array}\right\}$$

Medical diagnosis by CNs (a simple example of)

- Five Boolean vars
- Conditional independence relations by a DAG
- Elicitation of the local (conditional) CSs
- This is a CN specification
- The strong extension $K(S, C, B, X, D) =$



$P(s) \in [.25, .50]$

[EXE]
$\underline{P}(s, c, b, x, d) \; =?$

$P(c|s) \in [.15, .40]$
$P(c|\neg s) \in [.05, .10]$

$P(b|s) \in [.30, .55]$
$P(b|\neg s) \in [.20, .30]$

$P(x|c) \in [.90, .99]$
$P(x|\neg c) \in [.01, .05]$

$P(d|c, b) \in [.90, .99]$
$P(d|\neg c, b) \in [.50, .70]$
$P(d|c, \neg b) \in [.40, .60]$
$P(d|\neg c, \neg b) \in [.10, .20]$

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Updating with incomplete observations

- $\underline{P}(X_q = x_q | X_E = x_E, X_M = *)$
  $= \underline{P}(X_q = x_q | X_E = x_E)$
  right only if *missing at random* assumption holds
- *Conservative inference rule* (CIR)
  $\underline{P}(x_q | x_E, *) = \min_{x_M \in \Omega_{X_M}} P(x_q | x_E, x_M)$
  near-ignorance about the process preventing some variable from being observed *(de Cooman & Zaffalon, 2004)*
- CIR on CNs?
  - Add a (dummy) binary child for each missing, with vacuous quantification
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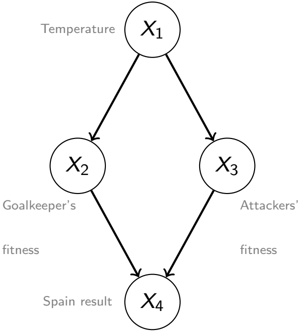

$\underline{P}(x_4 | X_1 = warm, X_2 = *)$

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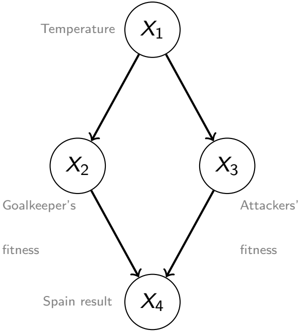

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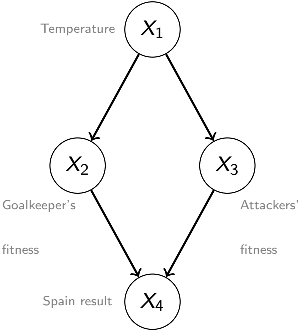

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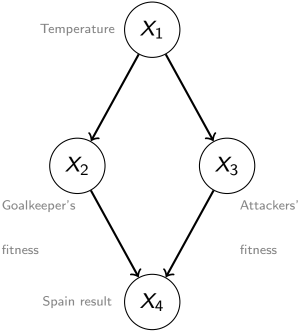

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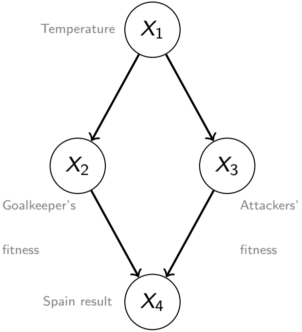

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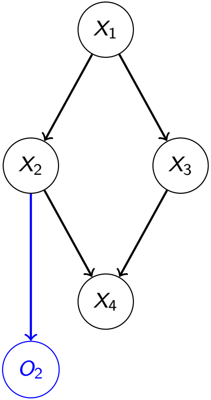

$\underline{K}(O_2|x_2)$ vacuous CS

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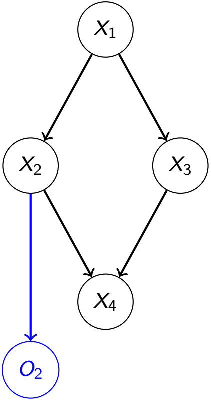

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Modelling the observational process

- Each X as a *latent* variable
- For each X a *manifest* variable O_X modelling the observation $\Omega_O = \Omega_X \cup \{*\}$
- Conditional independence, given X between O and the other variables (or weaker conditions)
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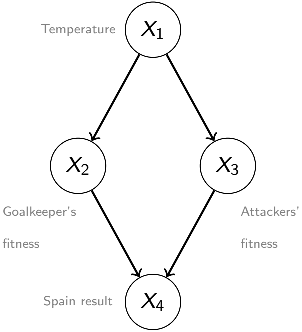

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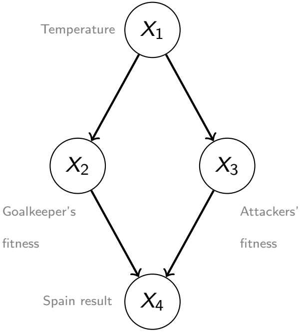

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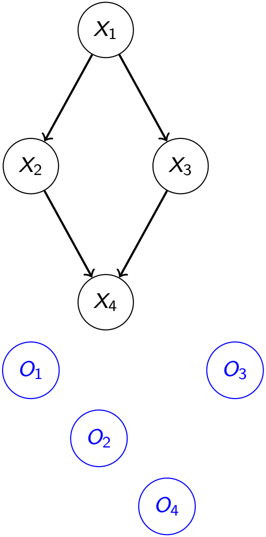

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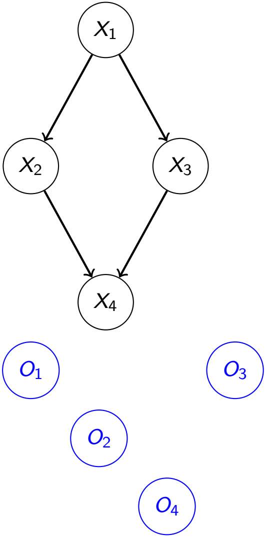

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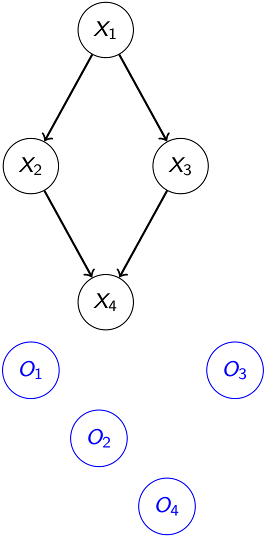

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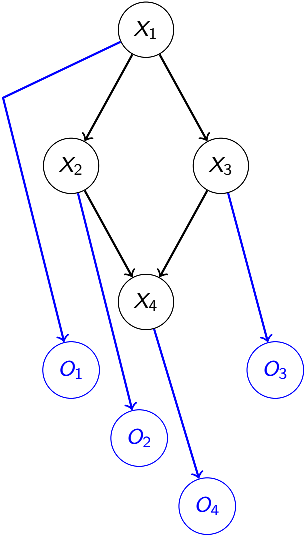

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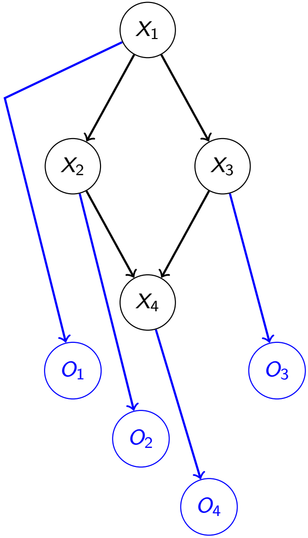

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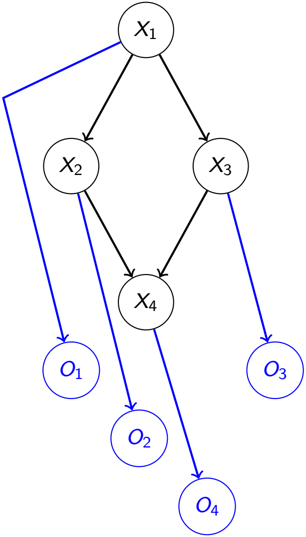

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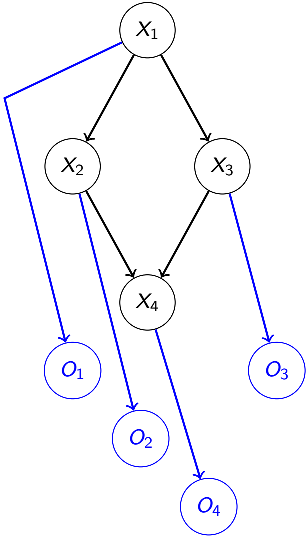

Modelling the observational process (ii)

- Manifest variables reduced to binary variables (coarsen to $\{o, \neg o\}$)
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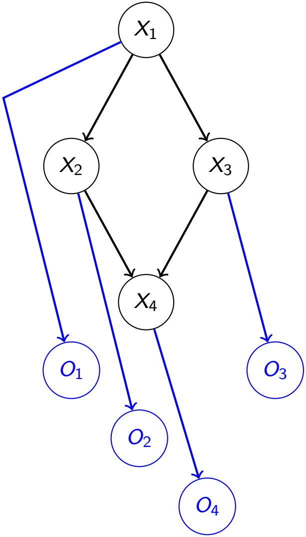

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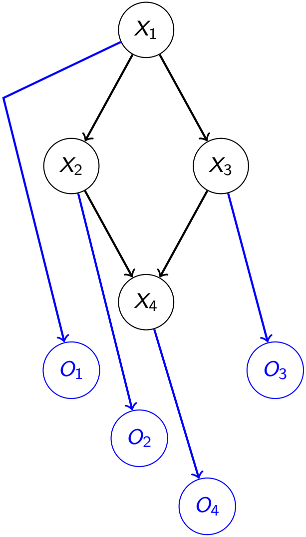

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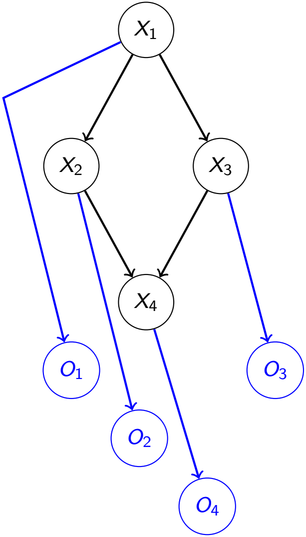

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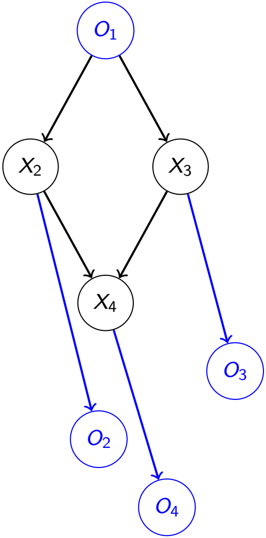

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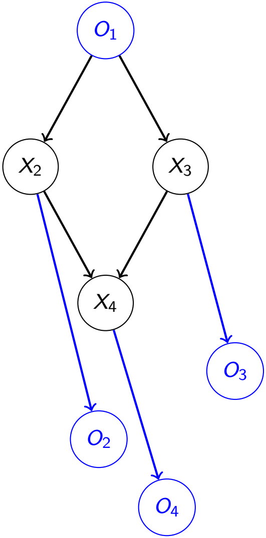

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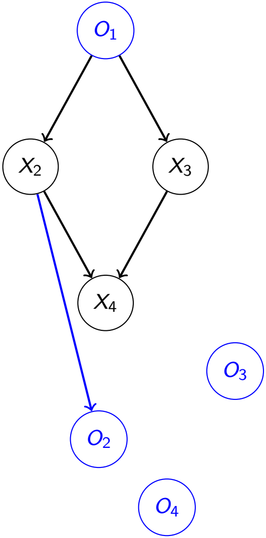

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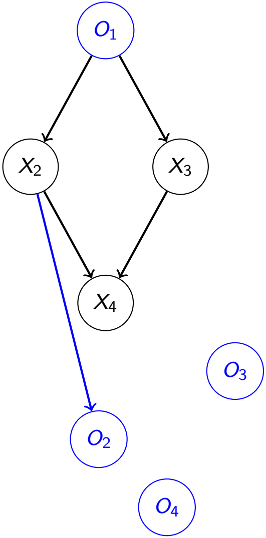

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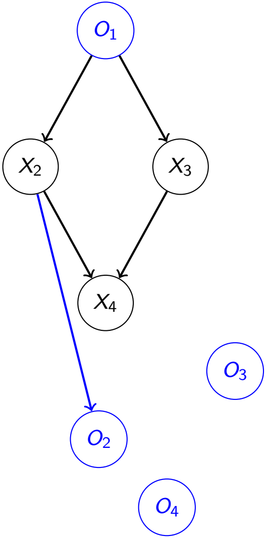

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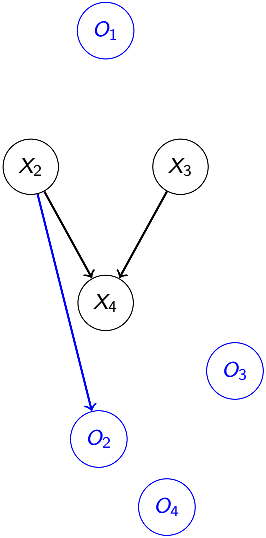
  - Imprecise likelihood ratio (and Jeffrey's rule)
- Hard evidence? Drop leaving arcs!
- Only the subnet connected to the query node



Modelling the observational process (ii)

- Manifest variables reduced to binary variables (coarsen to $\{o, \neg o\}$)
- Elicit only lower/upper likelihoods of observation given the latent $\underline{P}(o|x) \leq P(o|x) \leq \overline{P}(o|x)$
  - Perfect observation: $\underline{P}(o|x) = \overline{P}(o|x) = \delta_{o,x}$
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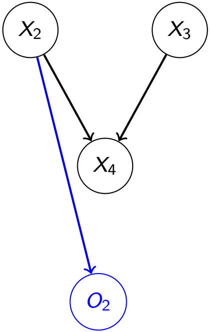

Averaging expert opinions

- Experts A and B give (precise knowledge) $P_A(X)$ and $P_B(X)$
- Arithmetic average $P_+(x) \propto P_A(x) \cdot P_B(x)$
- Geometric average $P_\times(x) \propto P_A(x) \cdot P_B(x)$
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Arithmetic average (Bayesian case)

Input $P_A(X)$, $P_B(X)$

Define $X_A, X_B : \Omega_{X_A} = \Omega_{X_B} = \Omega_X$

Two (independent) expert opinions

X is the averaged knowledge

D auxiliary var indexing experts: $\Omega_D = \{A, B\}$

$$P(x|x_A, x_B, d) = \begin{cases} \delta(x_a, x) & \text{if} \quad D = a \\ \delta(x_B, x) & \text{if} \quad D = b \end{cases}$$

Uniform prior: $P(D = A) = P(D = B)$

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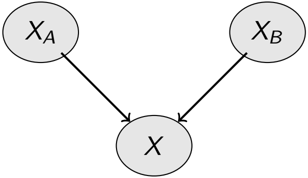

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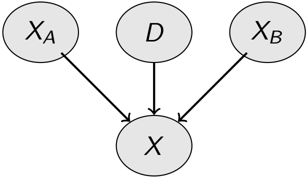

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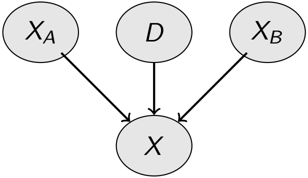

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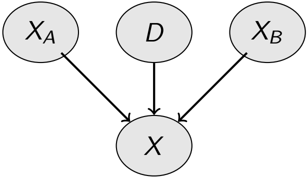

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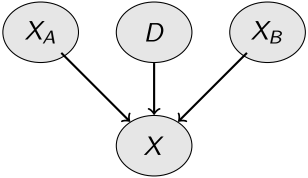

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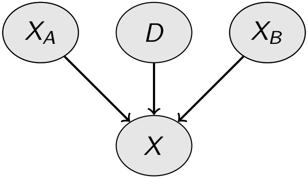
$P(D = A)$ and $P(D = B)$ as experts' reliabilities



$P(D = A) = 1 \Rightarrow P_+(X) = P_A(X)$

Prior ignorance $P(D = A) \in [0, 1]$ (vacuous)
$K_+(X) =$ convex closure of $\{P_A(X), P_B(X)\}$

Averaging imprecise expert knowledge,
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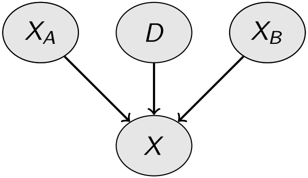

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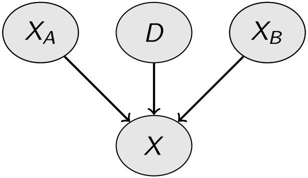

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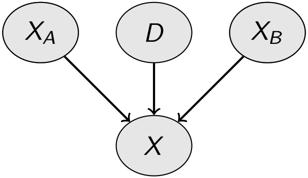

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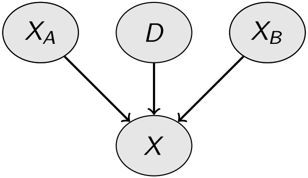

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Geometric average

Again two experts opinions $P(X_A) := P_A(X)$ and $P(X_B) := P_B(X)$

Modeling that X_a and X_B are the same variable

Logical constraints in BNs:
a dummy Boolean child D
true IFF $X_A = X_B$

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We can geometrically average credal sets!

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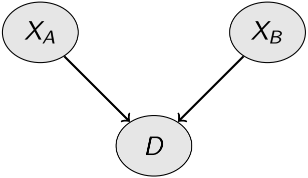

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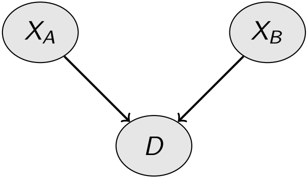

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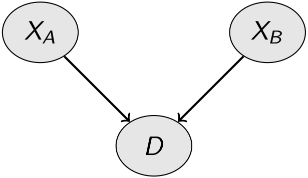

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Some experiments

- Tests with Boolean X
- Credal sets over Booleans are equivalent to intervals
  $P(X = T) \in [l, u] \Rightarrow K(X) = \left\{[l, 1 - l]^T, [u, 1 - u]^T\right\}$

$P(X_a = x)$	$P(X_b = x)$	$P(D = a)$	$P_+(x)$	$P_\times(x)$
.20	.40	.50	.30	.14
.20	.80	.50	.50	.50
[.10, .30]	[.30, .50]	.50	[.20, .40]	[.05, .30]
[.10, .30]	[.70, .90]	.50	[.40, .60]	[.21, .79]
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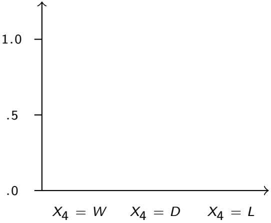
Decision Making on CNs

- Update beliefs about X_q (query)
  after the observation of x_E (evidence)
  What about the state of X_q?
- BN updating: compute $P(X_q|x_E)$
  State of X_q: $x_q^* = \arg\max_{x_q \in \Omega_{X_q}} P(x_q|x_E)$
- CN updating: compute $K(X_q|x_E)$?
  Algorithms only compute $\underline{P}(X_q|x_E)$
- State(s) of X_q by interval dominance
  $\Omega^*_{X_q} = \left\{ x_q \middle| \nexists x'_q \text{ s.t. } \underline{P}(x'_q|x_E) > \overline{P}(x_q|x_E) \right\}$
- More informative criterion: maximality
  $\left\{ x_q \middle| \nexists x'_q \text{ s.t. } P(x'_q|x_E) > P(x_q|x_E) \forall P(X_q|x_E) \in K(X_q|x_E) \right\}$
- Maximality can be computed with an auxiliary dummy child and standard updating (Antonucci & de Campos



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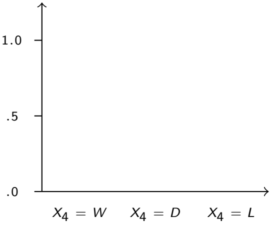

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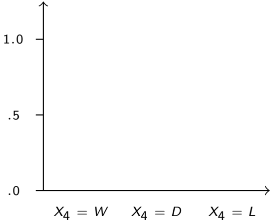

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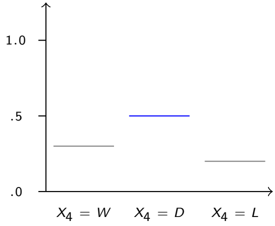

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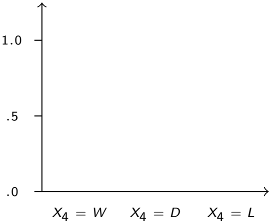

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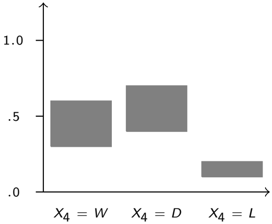

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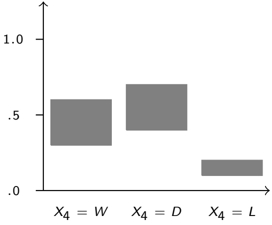

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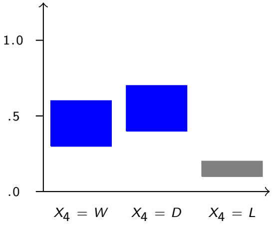

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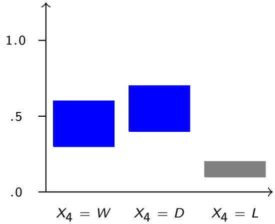

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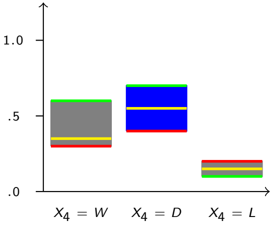

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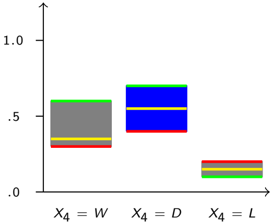

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A military application: no-fly zones surveillance

- Around important potential targets
  (eg. WEF, dams, nuke plants)
- Twofold circle wraps the target
  - External no-fly zone (sensors)
  - Internal no-fly zone (anti-air units)
- An aircraft entering the zone
  (to be called **intruder**)
- Its presence, speed, height, and other features revealed by the sensors
- A team of military experts decides:
  - what the intruder intends to do
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davos

dam

npp

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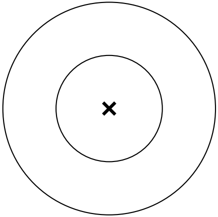

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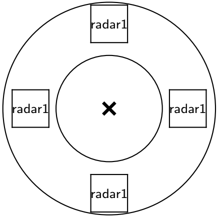

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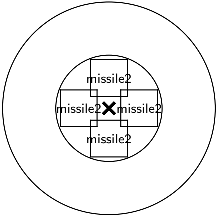

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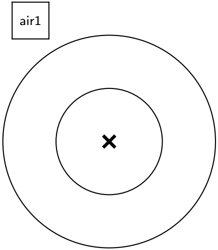

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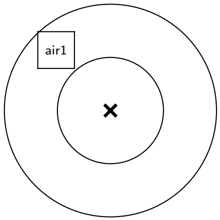

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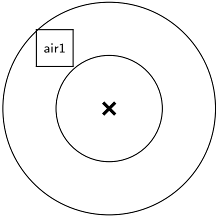

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- Four (exclusive and exhaustive) options for intruder's goal:

plane	rust	damaged	erroneous
renegade	provocateur	damaged	erroneous

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  - Sensors reliabilities are affected by geo/wheather conditions
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  - No deterministic relations between the different variables
  - Pervasive uncertainty in the observations
- Why a graphical model?
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- Why an imprecise (probabilistic) model?
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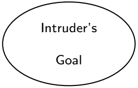

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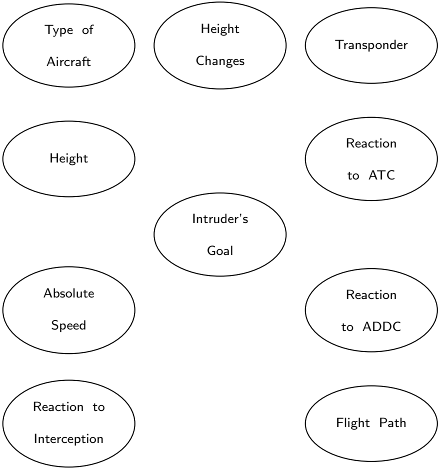

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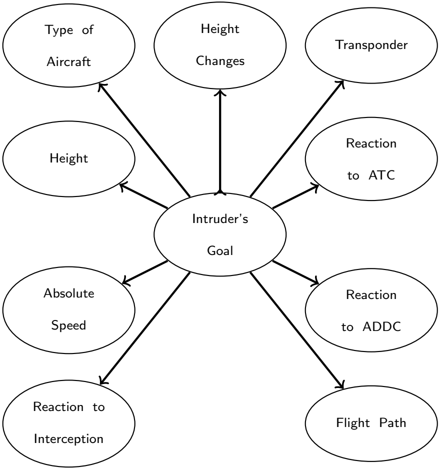

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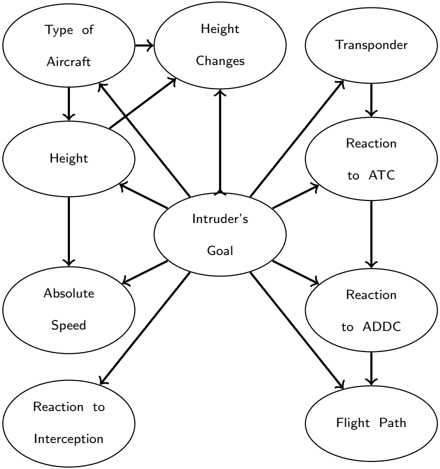

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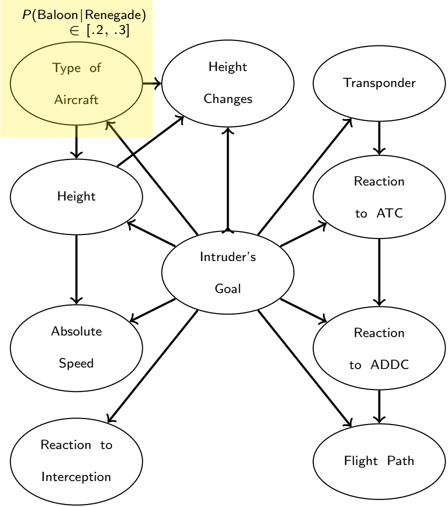

Network core

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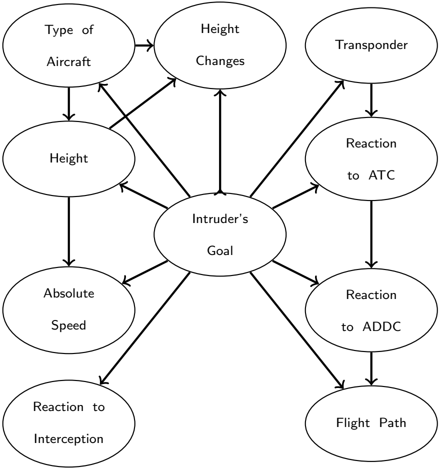

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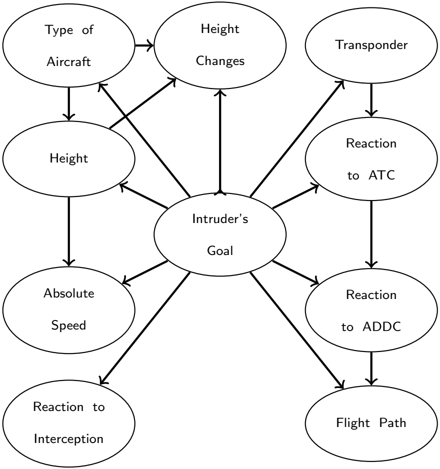
- Complex observation process!



Observations modelling and fusion

- Each sensor modeled by an auxiliary child of the (ideal) variable to be observed
- $P(\text{sensor}|\text{ideal})$ models sensor reliability

  (eg. identity matrix = perfectly reliable sensor)
- Many sensors? Many children!

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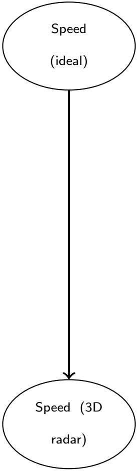

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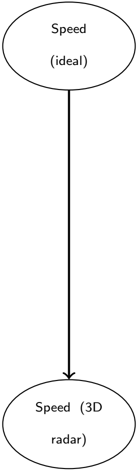

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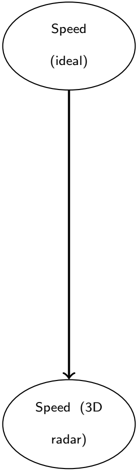

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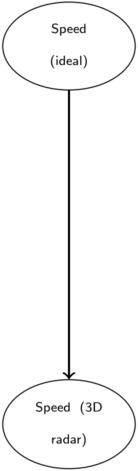

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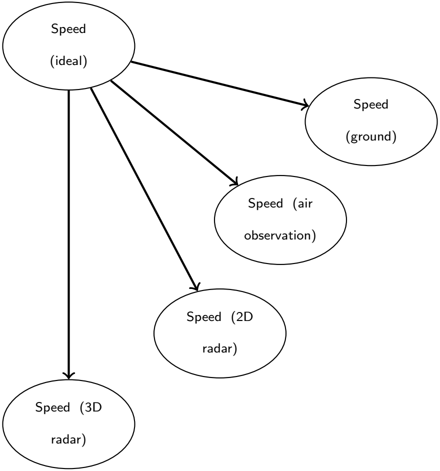

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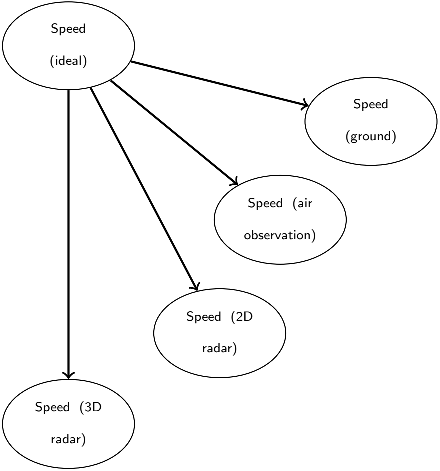
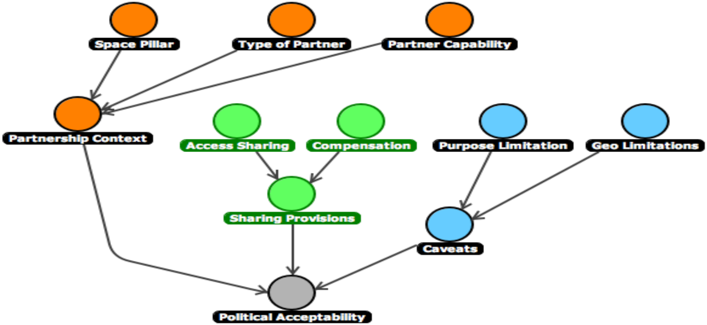


The whole network

- A huge multiply-connected credal network
- Efficient (approximate) updating with GL2U



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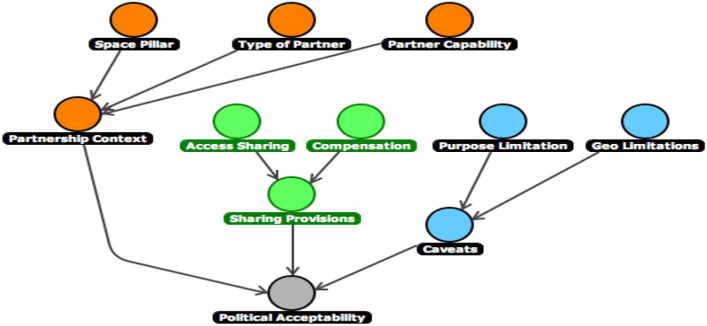

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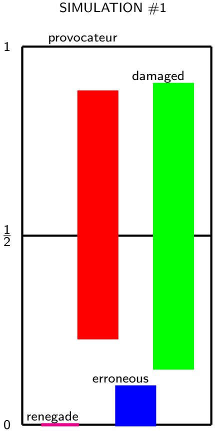

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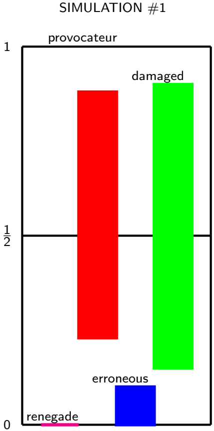

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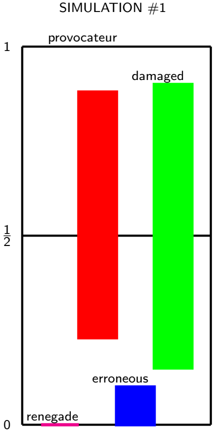

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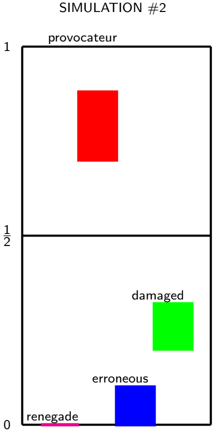

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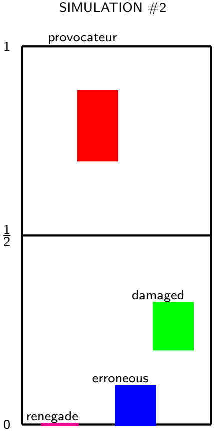
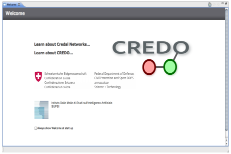


The CREDO software



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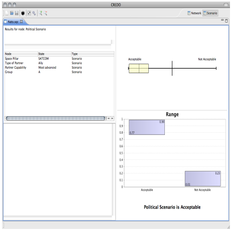

Results for node: Political Scenario

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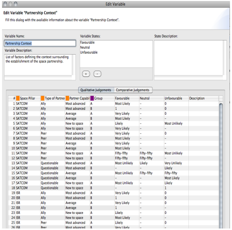

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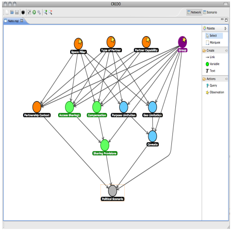

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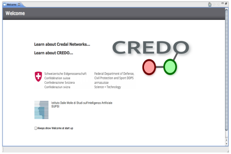

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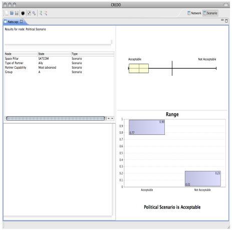

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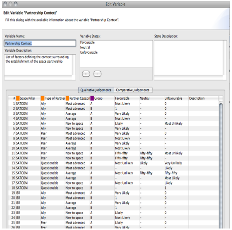

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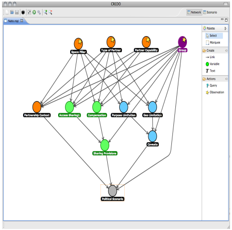

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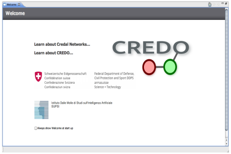

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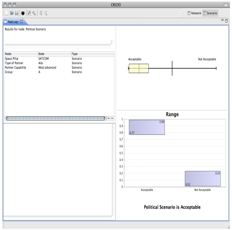

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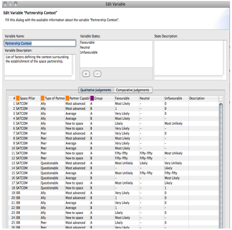

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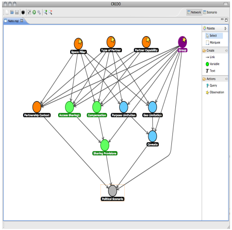

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Decision-Support System for Space Security

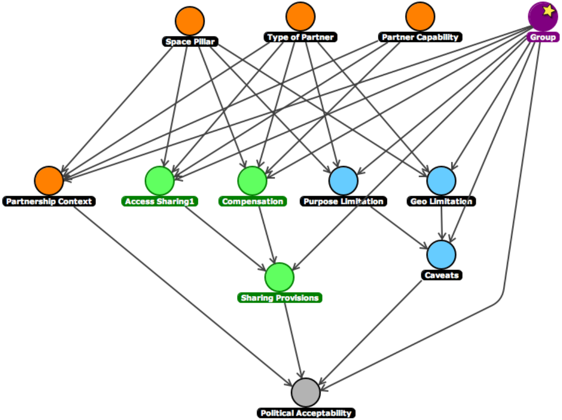
Variable of interest: *Political acceptability* (*acceptable* / *unacceptable*)

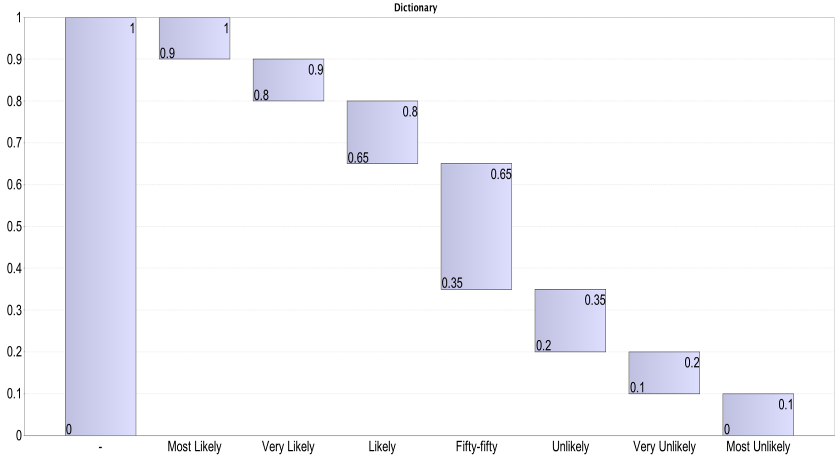
Observed features

- *Space pillar*: possible states *SATCOM* (command, control, communication and computer systems dependent on satellites communication), *ISR* (synchronized and integrated planning) and *SSA* (ability to obtain information and knowledge about the space beyond the Earth atmosphere).
- *Type of partner*: *ally*, *peer* and *questionable*.
- *Partner capability*: *most advanced*, *average* and *new to space*.

Intermediate variables

- *Access sharing*: *C2 payload and raw data* (ability to directly manage the beam), *raw data only* and *no direct access*.
- *Compensation*: *in-kind*, *small*, *medium* and *large compensation*.
- *Purpose limitation*: *peaceful and non-economic*, *peaceful only* and *no limitations*.
- *Geographical limitation*: *peace-keeping exclusion*, *partner exclusion* and *no limitations*.


Space Pillar
Type of Partner
Partner Capability
Group
Partnership Context
Access Sharing1
Compensation
Purpose Limitation
Geo Limitation
Caveats
Sharing Provisions
Political Acceptability


Dictionary
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-
0
1
Most Likely
0.9
1
Very Likely
0.8
0.9
Likely
0.65
0.8
Fifty-fifty
0.35
0.65
Unlikely
0.2
0.35
Very Unlikely
0.1
0.2
Most Unlikely
0
0.1

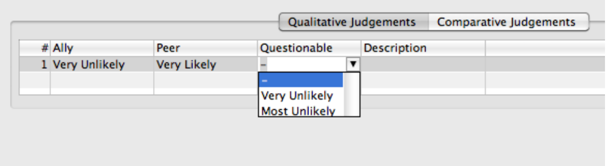
Preventing inconsistent judgements

- E.g., two states of X cannot be both "likely" (as this means $P(x) > .65$, $\sum_x P(x) > 1$).
- Reachability constraints

$$\sum_{x \in \Omega_X \setminus \{x'\}} \underline{P}(x) + \overline{P}(x') \quad \leq 1, \tag{1}$$

$$\sum_{x \in \Omega_X \setminus \{x'\}} \overline{P}(x) + \underline{P}(x') \quad \geq 1. \tag{2}$$

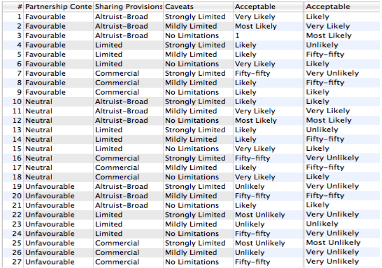
- Judgement specification is sequential, the software displays only consistent options



#	Ally	Peer	Questionable	Description
1	Very Unlikely	Very Likely	-	

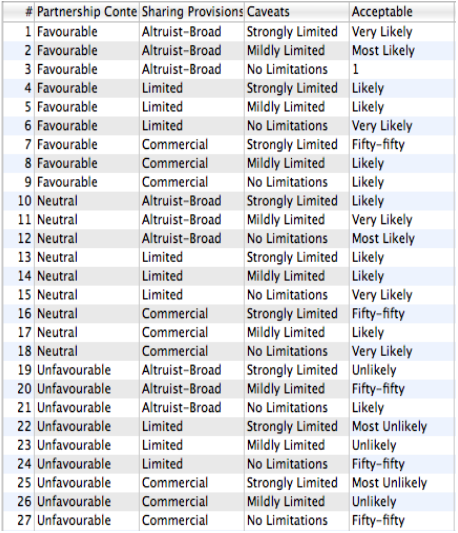
NATO Multinational Experiment 7

- Concerned with protecting our access to the global commons.
- During the final meeting a group of six subject matter experts (divided into two groups) developed its own conclusions about political acceptability for 27 scenarios
- Human experts reasoning vs. (almost) automatic reasoning with credal networks (quantified by expert knowledge)



#	Partnership Conte	Sharing Provisions	Caveats	Acceptable	Acceptable
1	Favourable	Altruist-Broad	Strongly Limited	Very Likely	Likely
2	Favourable	Altruist-Broad	Mildly Limited	Most Likely	Very Likely
3	Favourable	Altruist-Broad	No Limitations	1	Most Likely
4	Favourable	Limited	Strongly Limited	Likely	Unlikely
5	Favourable	Limited	Mildly Limited	Likely	Fifty-fifty
6	Favourable	Limited	No Limitations	Very Likely	Likely
7	Favourable	Commercial	Strongly Limited	Fifty-fifty	Very Unlikely
8	Favourable	Commercial	Mildly Limited	Likely	Fifty-fifty
9	Favourable	Commercial	No Limitations	Likely	Likely
10	Neutral	Altruist-Broad	Strongly Limited	Likely	Likely
11	Neutral	Altruist-Broad	Mildly Limited	Very Likely	Very Likely
12	Neutral	Altruist-Broad	No Limitations	Most Likely	Most Likely
13	Neutral	Limited	Strongly Limited	Likely	Unlikely
14	Neutral	Limited	Mildly Limited	Likely	Fifty-fifty
15	Neutral	Limited	No Limitations	Very Likely	Likely
16	Neutral	Commercial	Strongly Limited	Fifty-fifty	Very Unlikely
17	Neutral	Commercial	Mildly Limited	Likely	Fifty-fifty
18	Neutral	Commercial	No Limitations	Very Likely	Likely
19	Unfavourable	Altruist-Broad	Strongly Limited	Unlikely	Very Unlikely
20	Unfavourable	Altruist-Broad	Mildly Limited	Fifty-fifty	Fifty-fifty
21	Unfavourable	Altruist-Broad	No Limitations	Likely	Likely
22	Unfavourable	Limited	Strongly Limited	Most Unlikely	Very Unlikely
23	Unfavourable	Limited	Mildly Limited	Unlikely	Unlikely
24	Unfavourable	Limited	No Limitations	Fifty-fifty	Very Unlikely
25	Unfavourable	Commercial	Strongly Limited	Most Unlikely	Most Unlikely
26	Unfavourable	Commercial	Mildly Limited	Unlikely	Very Unlikely
27	Unfavourable	Commercial	No Limitations	Fifty-fifty	Very Unlikely

Group A



#	Partnership Conte	Sharing Provisions	Caveats	Acceptable
1	Favourable	Altruist-Broad	Strongly Limited	Very Likely
2	Favourable	Altruist-Broad	Mildly Limited	Most Likely
3	Favourable	Altruist-Broad	No Limitations	1
4	Favourable	Limited	Strongly Limited	Likely
5	Favourable	Limited	Mildly Limited	Likely
6	Favourable	Limited	No Limitations	Very Likely
7	Favourable	Commercial	Strongly Limited	Fifty-fifty
8	Favourable	Commercial	Mildly Limited	Likely
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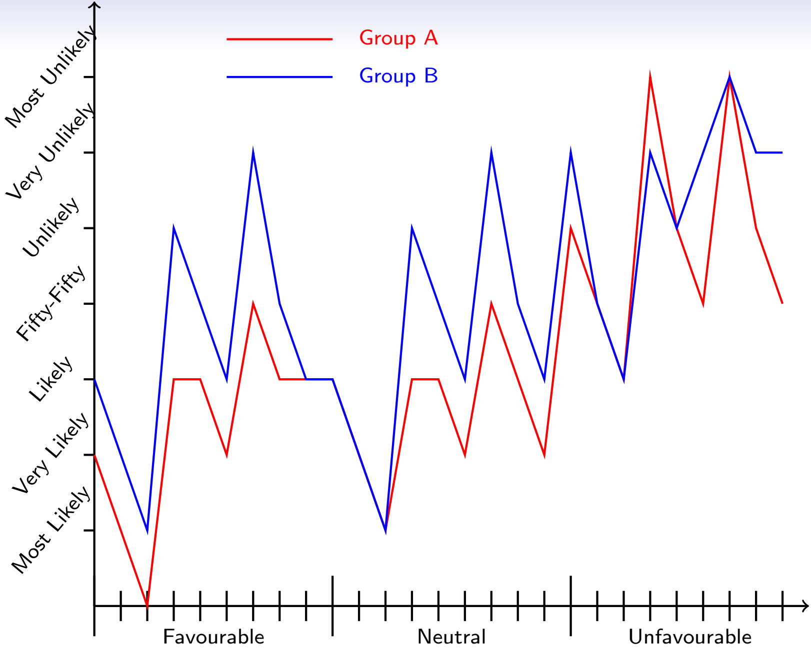
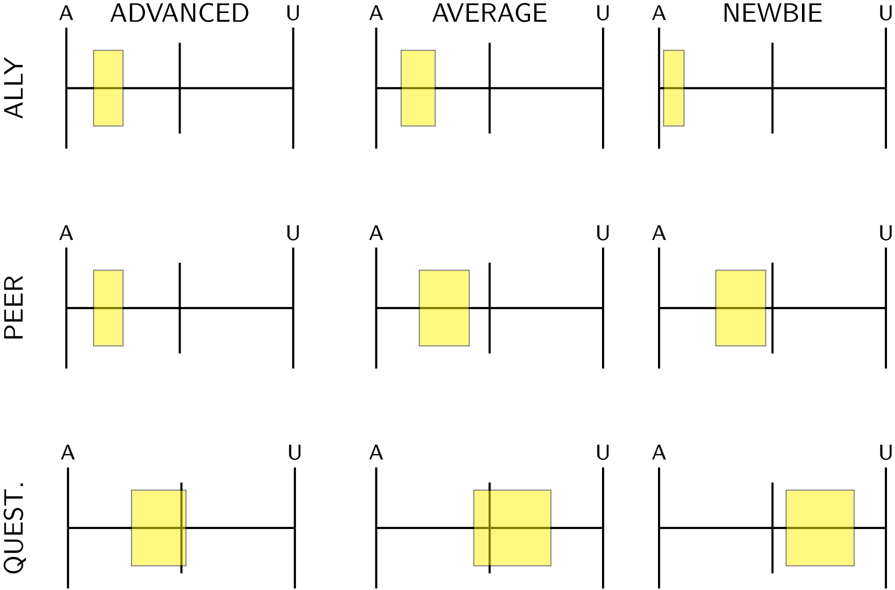
10	Neutral	Altruist-Broad	Strongly Limited	Likely
11	Neutral	Altruist-Broad	Mildly Limited	Very Likely
12	Neutral	Altruist-Broad	No Limitations	Most Likely
13	Neutral	Limited	Strongly Limited	Likely
14	Neutral	Limited	Mildly Limited	Likely
15	Neutral	Limited	No Limitations	Very Likely
16	Neutral	Commercial	Strongly Limited	Fifty-fifty
17	Neutral	Commercial	Mildly Limited	Likely
18	Neutral	Commercial	No Limitations	Very Likely
19	Unfavourable	Altruist-Broad	Strongly Limited	Unlikely
20	Unfavourable	Altruist-Broad	Mildly Limited	Fifty-fifty
21	Unfavourable	Altruist-Broad	No Limitations	Likely
22	Unfavourable	Limited	Strongly Limited	Most Unlikely
23	Unfavourable	Limited	Mildly Limited	Unlikely
24	Unfavourable	Limited	No Limitations	Fifty-fifty
25	Unfavourable	Commercial	Strongly Limited	Most Unlikely
26	Unfavourable	Commercial	Mildly Limited	Unlikely
27	Unfavourable	Commercial	No Limitations	Fifty-fifty

Group B

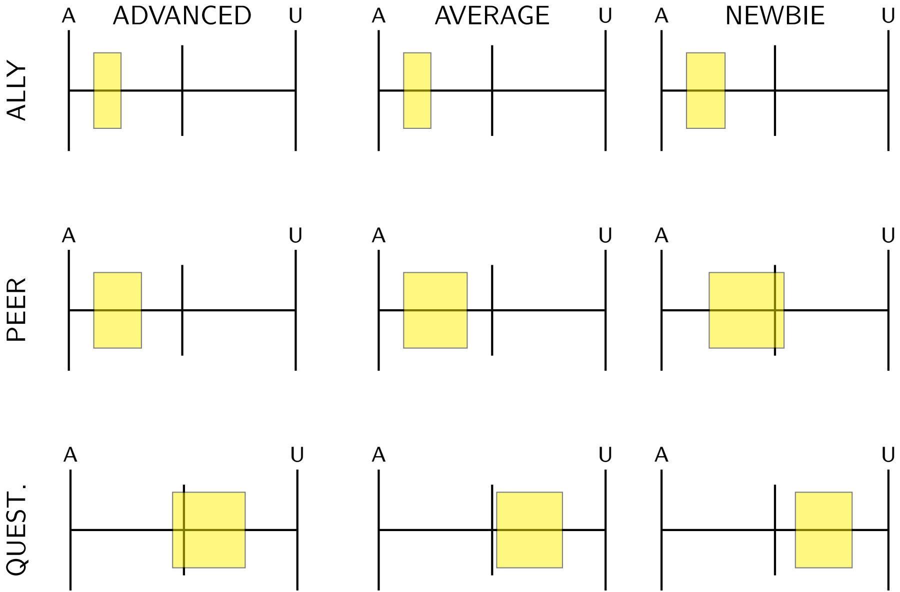
Acceptable
Likely
Very Likely
Most Likely
Unlikely
Fifty-fifty
Likely
Very Unlikely
Fifty-fifty
Likely
Likely
Very Likely
Most Likely
Unlikely
Fifty-fifty
Likely
Very Unlikely
Fifty-fifty
Likely
Very Unlikely
Fifty-fifty
Likely
Very Unlikely
Unlikely
Very Unlikely
Most Unlikely
Very Unlikely
Very Unlikely


Group A
Group B
Most Unlikely
Very Unlikely
Unlikely
Fifty-Fifty
Likely
Very Likely
Most Likely
Favourable
Neutral
Unfavourable

ISR (Group A)



ISR (Group B)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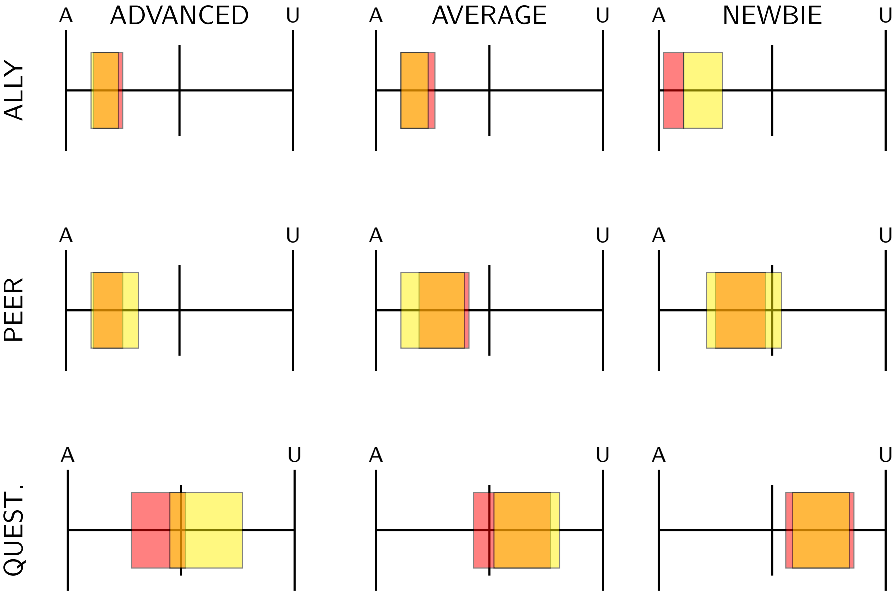
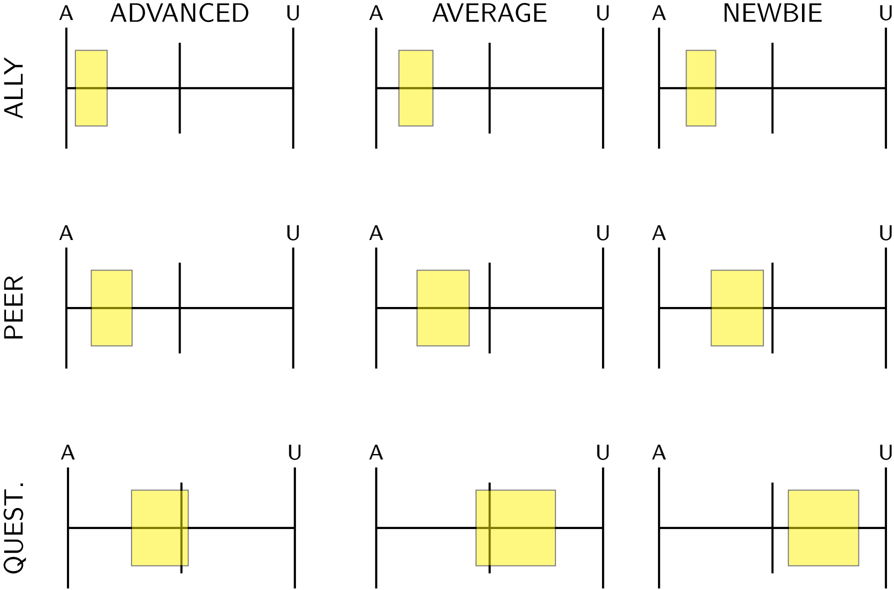
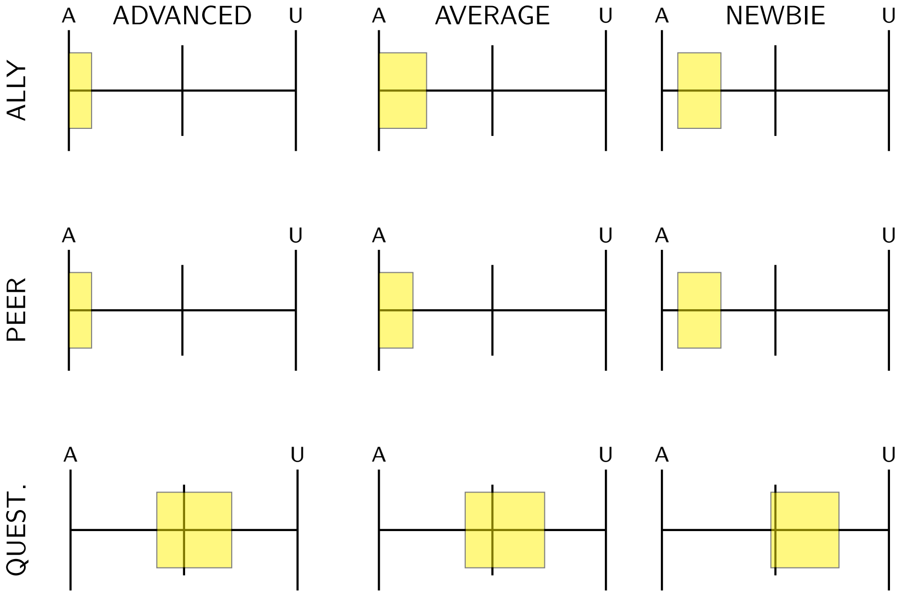
ISR (Group A+B)



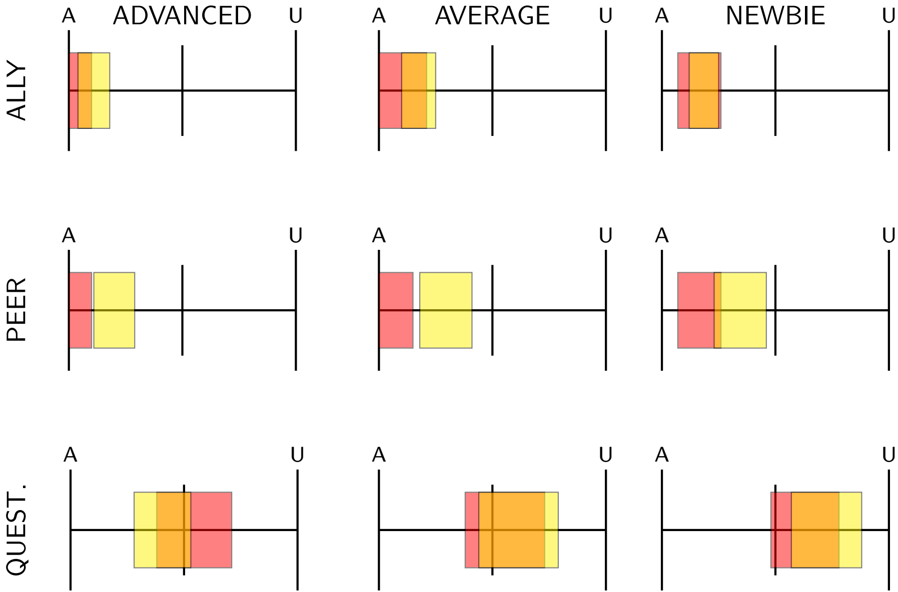
SSA (Group A)



SSA (Group B)



SSA (Group A+B)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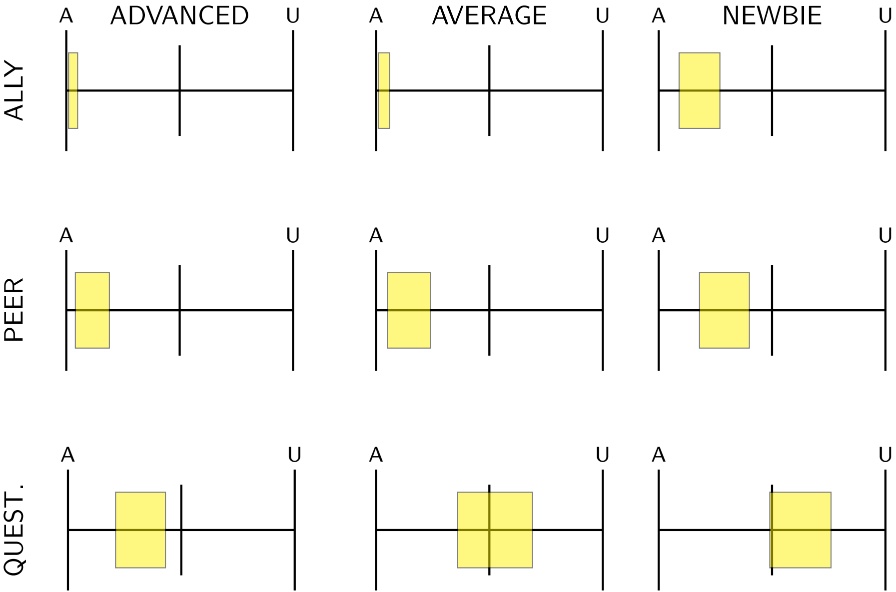
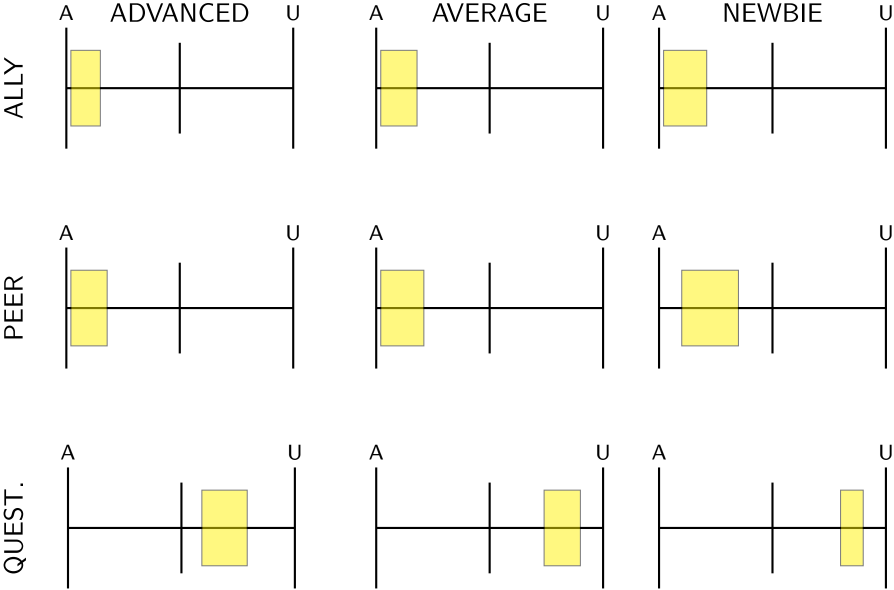
QUEST.

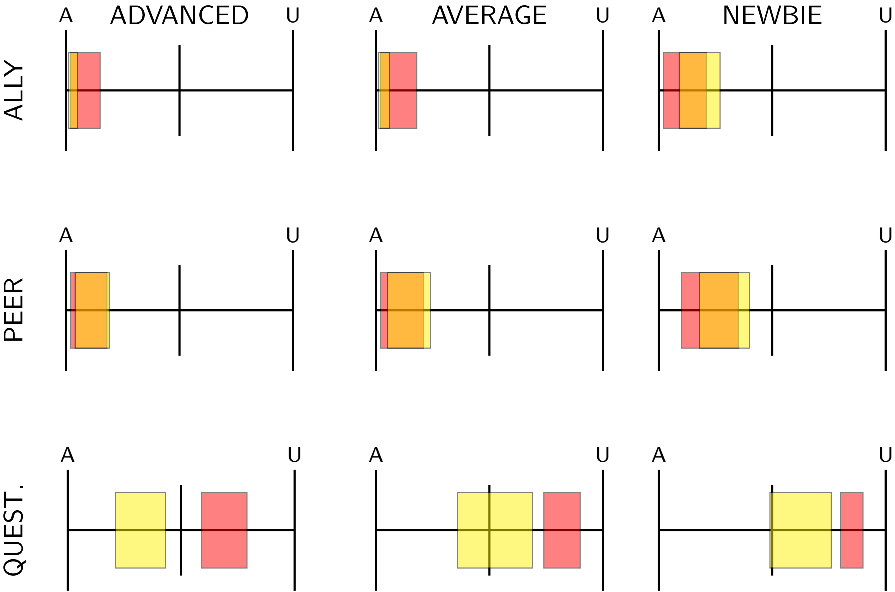
SATCOM (Group A)



SATCOM (Group B)



SATCOM (Group A+B)



Experiment conclusions

- 27 vignettes
- A and B agree on a single answer 18
- A and B agree on suspending judgement 3
- A suspend , B not or vice versa 5
- A and B disagree 1
- Good agreement, not-too-imprecise outputs, results consistent with human conclusions

Environmental example: debris flows hazard assessment

debrisflows

df

- Debris flows are very destructive natural hazards
- Still partially understood
- Human expertize is still fundamental!
- An artificial expert system supporting human experts?

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Causal modelling

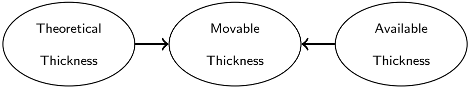
Proxy indicator of the level of risk

Movable

Thickness

Causal modelling

Triggering Factors

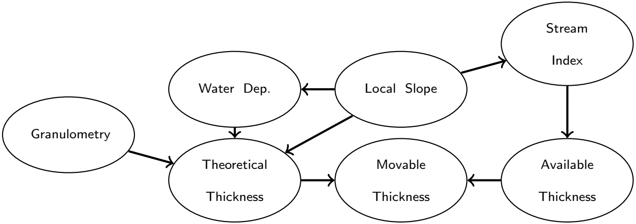


Theoretical Thickness

Movable Thickness

Available Thickness

Causal modelling



Stream Index

Water Dep.

Local Slope

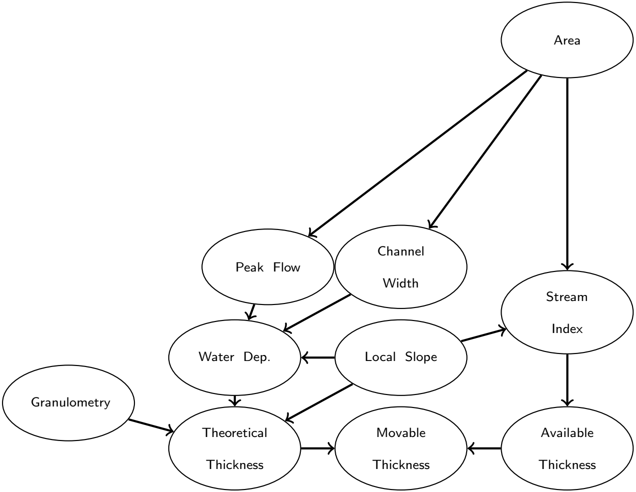
Granulometry

Theoretical Thickness

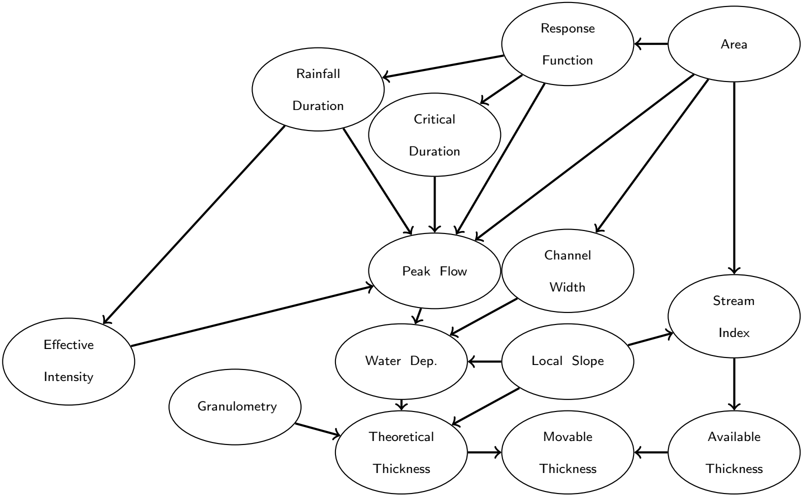
Movable Thickness

Available Thickness

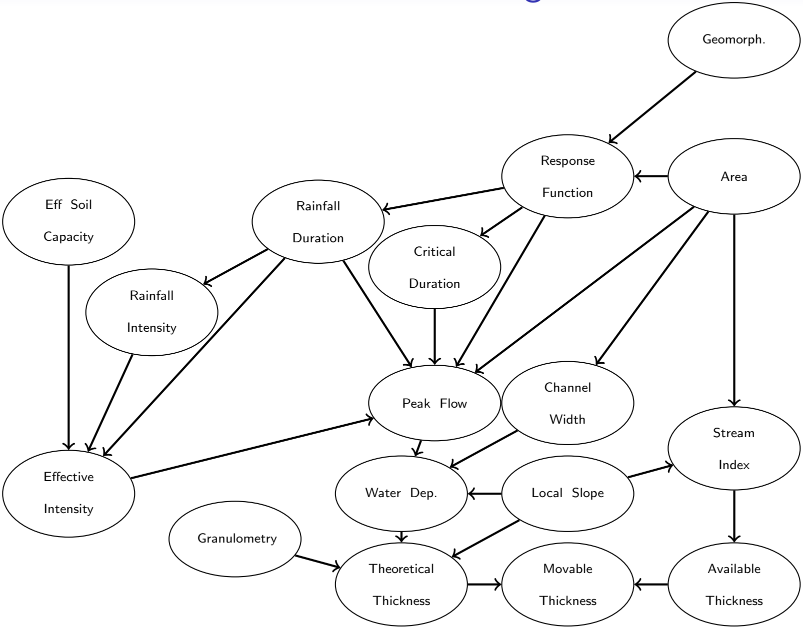
Causal modelling



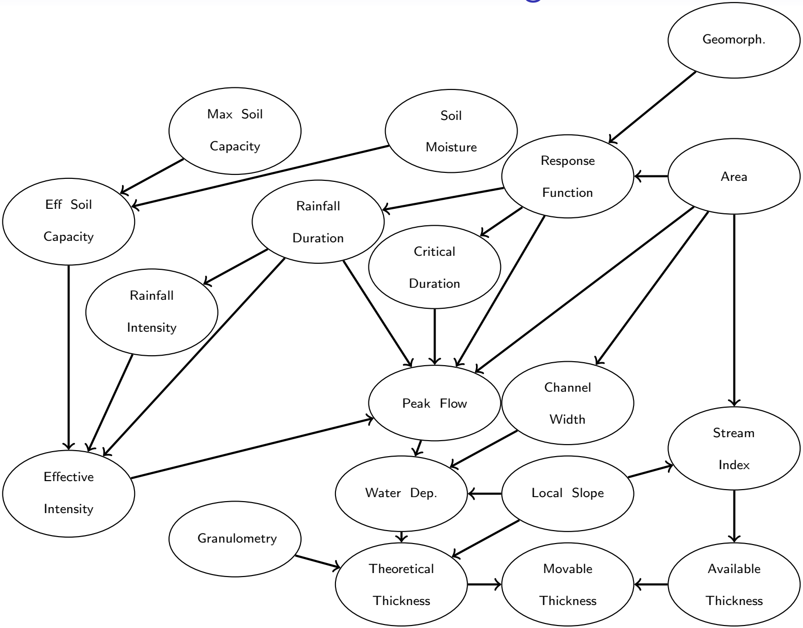
Causal modelling



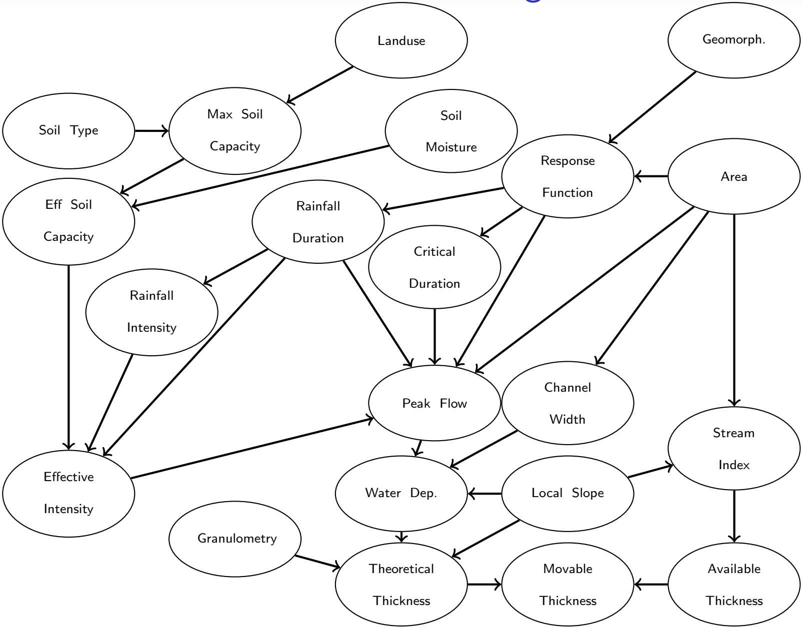
Causal modelling



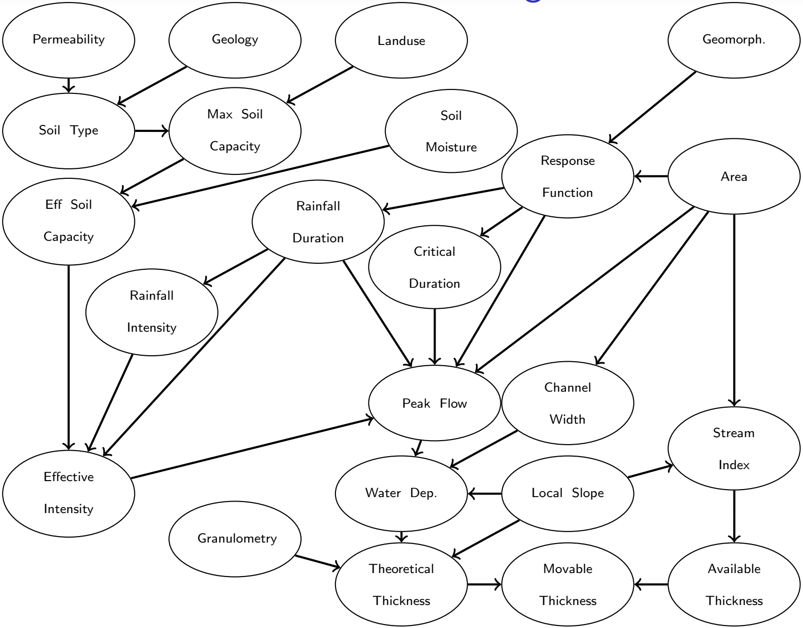
Causal modelling



Causal modelling

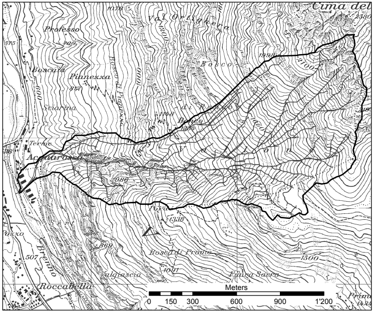


Causal modelling



Debris flow hazard assessment by CNs

- Extensive simulations in a debris flow prone watershed Acquarossa Creek Basin (area 1.6 Km^2, length 3.1 Km)



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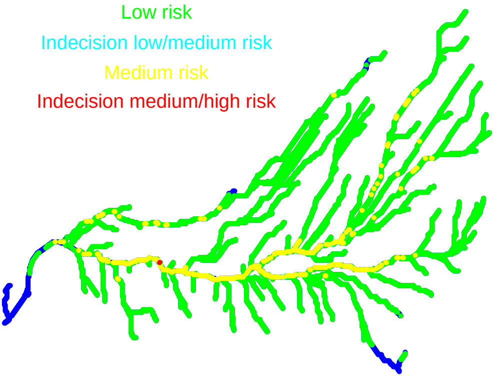

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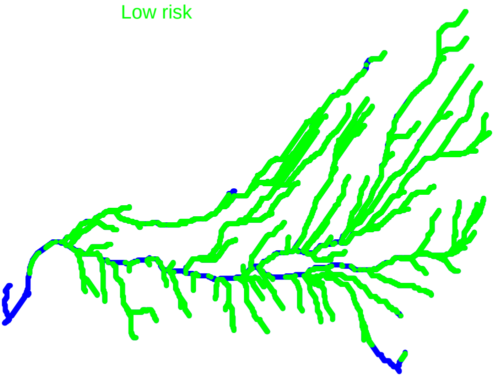

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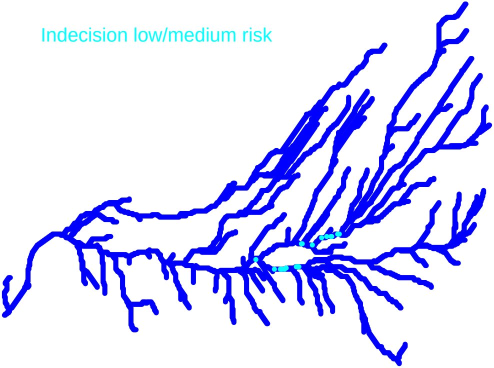

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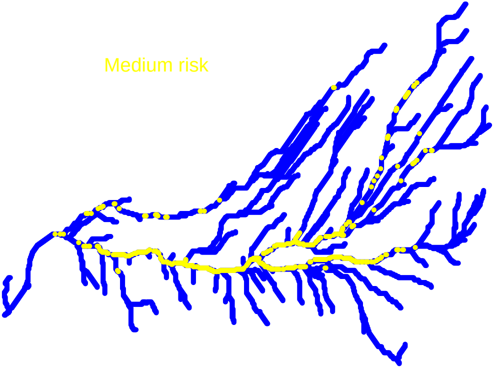

Debris flow hazard assessment by CNs

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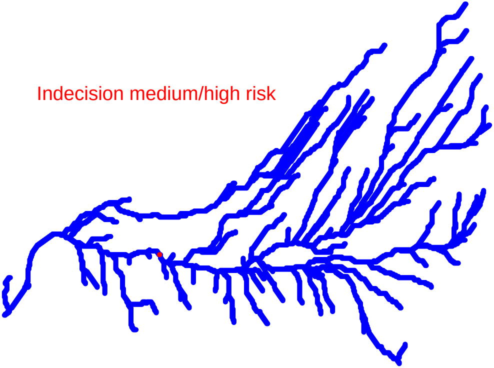
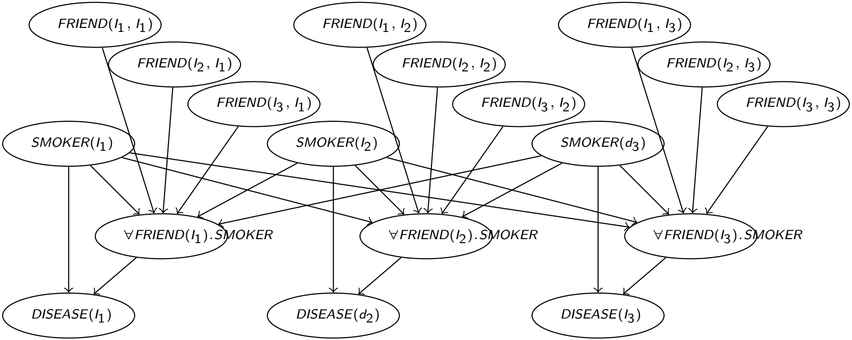


$\mathcal{CRALC}$ probabilistic logic with IPs *(Cozman, 2008)*

- Description logic with interval of probabilities
- N individuals $(I_1, \ldots, I_n)$,
  $P(smoker(I_i)) \in [.3, .5],\ P(friend(I_j, I_i)) \in [.0, .5]$,
  $P(disease(I_i)|smoker(I_i), \forall friend(I_j, I_i).I_i smoker) = \ldots$
- $\underline{P}(disease)$? Inference $\equiv$ updating of a (large) binary CN



References

- Piatti, A., Antonucci, A., Zaffalon, M. (2010). Building knowledge-based expert systems by credal networks: a tutorial. In Baswell, A.R. (Ed), Advances in Mathematics Research 11, Nova Science Publishers, New York.
- Corani, G., Antonucci, A., Zaffalon, M. (2012). Bayesian networks with imprecise probabilities: theory and application to classification. In Holmes, D.E., Jain, L.C. (Eds), Data Mining: Foundations and Intelligent Paradigms, Intelligent Systems Reference Library 23, Springer, Berlin / Heidelberg, pp. 49–93.
- `ipg.idsia.ch`
- `www.sipta.org`