COMPUTING THE OPTIMAL GAME

… and implications for possible and necessary winners of tournament solutions

Markus Brill (University of Oxford)

Joint work with **Rupert Freeman** & **Vincent Conitzer**

LABEX CIMI Pluridisciplinary Workshop on Game Theory

Example

	L	*R*
T	1	2
B	**{0,2}**	3

	1 *L*	*R*
1 *T*	1	2
B	***0***	3

	1 *L*	*R*
T	1	2
1 *B*	***2***	3

Equilibrium Actions (EAs):

- *L* is a *necessary* EA
- *T* and *B* are *possible* EAs
- *R* is not a possible EA

Motivation

- Game designer with limited control
  - between game theory (no control) and mechanism design (full control)
- Also captures settings without designer
  - settings with (non-probabilistic) uncertainty about payoffs
  - empirical game theory: expensive to determine payoffs
- Social choice functions defined via games
  - bipartisan set (BP): equilibrium actions of tournament game based on pairwise comparisons of alternatives
  - possible/necessary BP winners = poss./nec. equilibrium actions

Setting

- Two-player zero-sum games (aka matrix games)
  - incomplete matrix games: some payoffs given by sets
  - completion: pick one element from each set
- We are interested in Nash equilibria of completions
  - possible equilibrium action: action played with positive probability in *some* completion
  - necessary equilibrium action: action played with positive probability in *all* completions
- To avoid multiplicity issues: quasi-strict equilibrium [Harsanyi 1973]
  - unique support in matrix games [Brandt & Fischer 2008]
  - support consists of all actions that are played in some Nash equilibrium

Outline

- Matrix Games
  - greedy algorithm
  - complexity results
- Tournament Games
  - bipartisan set
  - complexity results
  - MIP formulation
- Future Directions

Greedy Algorithm for Matrix Games

- Idea: Check whether action a^* is a possible EA by only considering the extension that maximizes $u(a^*,\cdot)$ and minimizes $u(b,\cdot)$ for $b \neq a^*$.
- Does *not* work:

	L	*R*
T	*0*	*3*
B	**{1,3}**	*2*

	L (1)	*R*
T	*0*	*3*
(1) *B*	***1***	*2*

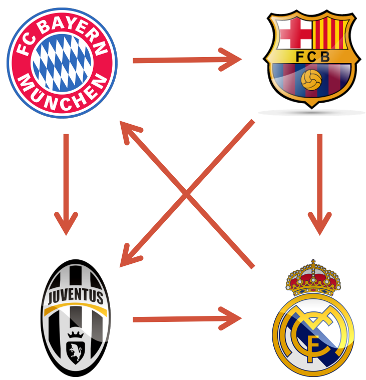
	L (1/4)	*R* (3/4)
(1/4) *T*	*0*	*3*
(3/4) *B*	***3***	*2*

Hardness Results for Matrix Games

- **Theorem.** Computing possible equilibrium actions of an incomplete matrix game is NP-complete.
- **Theorem.** Computing necessary equilibrium actions of an incomplete matrix game is coNP-complete.
  - Proof: Reduction from SetCover

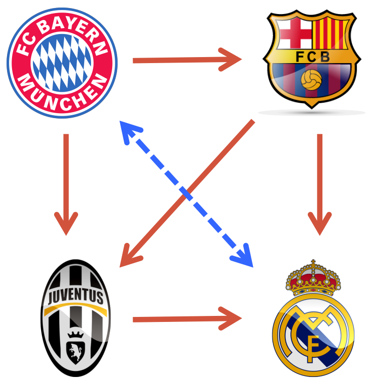
	c_1	c_2	c_3	c_4	c_5	s_1	s_2	s_3	s_4	s_5	t
$S_{1,1}$	0	H	$-H$	H	$-H$	B	B	0	0	0	N_1
$S_{1,2}$	$\{-1,1\}$	H	$-H$	H	$-H$	-1	-1	0	0	0	N_2
$S_{2,1}$	$-H$	0	H	$-H$	H	0	0	B	B	0	N_1
$S_{2,2}$	$-H$	$\{-1,1\}$	H	$-H$	H	0	0	-1	-1	0	N_2
$S_{3,1}$	H	$-H$	0	H	$-H$	0	0	B	0	B	N_1
$S_{3,2}$	H	$-H$	$\{-1,1\}$	H	$-H$	0	0	-1	0	-1	N_2
$S_{4,1}$	$-H$	H	$-H$	0	H	0	B	0	B	0	N_1
$S_{4,2}$	$-H$	H	$-H$	$\{-1,1\}$	H	0	-1	0	-1	0	N_2
x_1	H	$-H$	H	H	-1	0	0	0	0	0	0

Tournament Games



	0	1	1	-1
	-1	0	1	1
	-1	-1	0	1
	1	-1	-1	0

Tournament Games



	0	1	1	{-1,1}
	-1	0	1	1
	-1	-1	0	1
	{-1,1}	-1	-1	0

Tournament Games

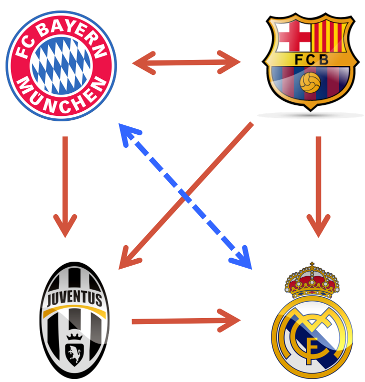
0	1	1	{-1,1}
-1	0	1	1
-1	-1	0	1
{-1,1}	-1	-1	0

	1			
1	0	1	1	1
	-1	0	1	1
	-1	-1	0	1
	-1	-1	-1	0

	1/3	1/3		1/3
1/3	0	1	1	-1
1/3	-1	0	1	1
	-1	-1	0	1
1/3	1	-1	-1	0

Hardness Results for Tournaments

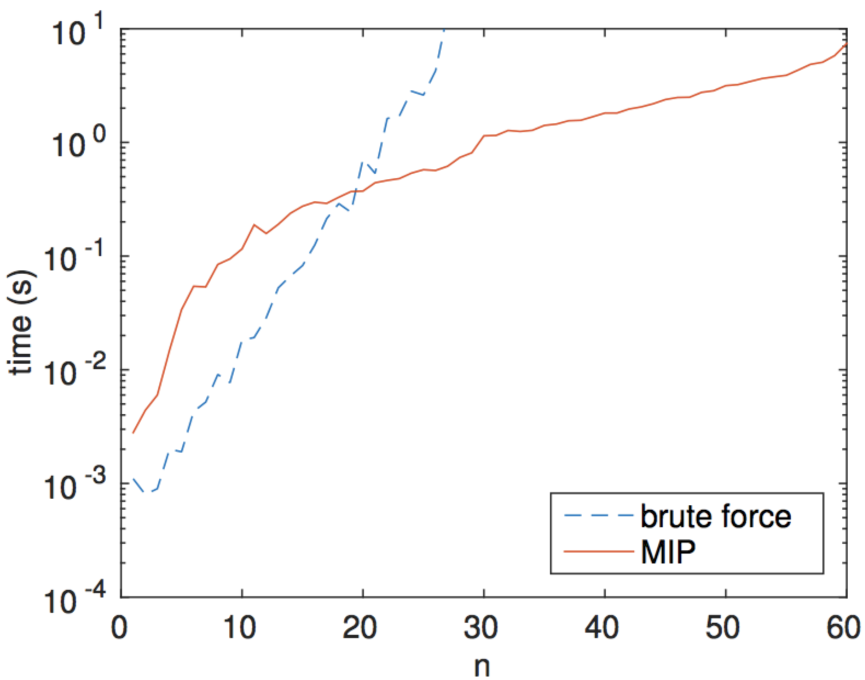
- Bipartisan set (BP) of a tournament game: all alternatives that are played in equilibrium [Laffond, Laslier, & Le Breton 1993]
  - thus: pos./nec. equilibrium actions = pos./nec. BP winners
- **Theorem.** Computing possible (necessary) BP winners of incomplete tournament games is (co)NP-complete.
  - Proof: reduction from 3SAT. … cyclones! … components!! … local reversal!!!
- **Theorem.** In weak tournament games (with ties *and* payoff sets {-1,0,1}), computing possible & necessary equilibrium actions is NP-hard.
  - hardness even holds for continuous payoff sets [-1,1]
  - equilibrium actions of a weak tournament game: "essential set" [Dutta & Laslier 1999]

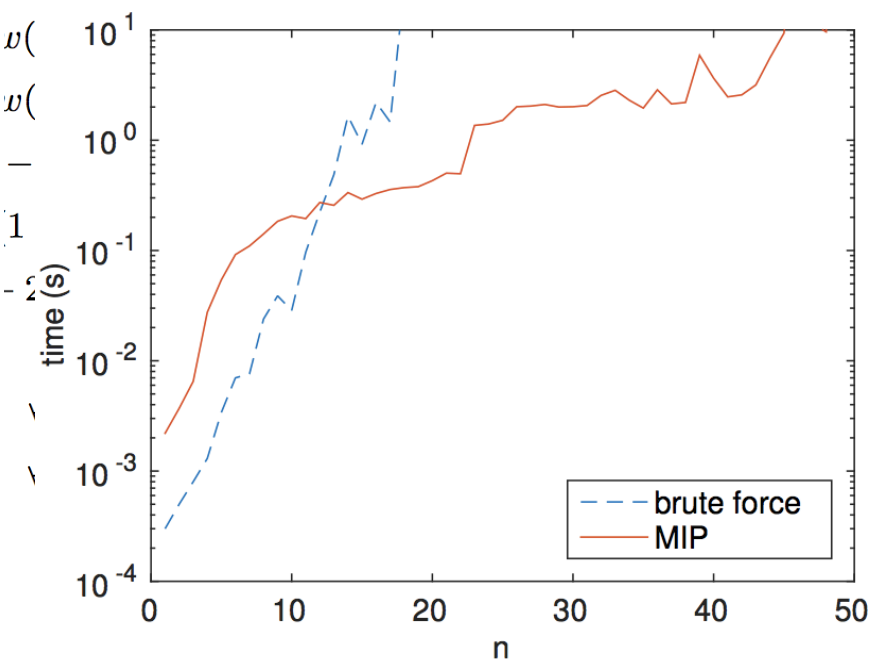


MIP Formulation for Tournament Games

- We formulated the possible equilibrium action problem for (weak) tournament games as a mixed integer linear program
- Runtime comp
  - left: n/2 unspec d entries (n = #alternatives)

$$\text{maximize} \quad p_k$$
$$\text{subject to} \quad x_{ij}^{\text{neg}} - x_{ji}^{\text{pos}} = 0, \quad \forall i, j$$
$$x^{\text{pos}} + x^{\text{neg}} \le 1, \quad \forall i, j$$





Future Directions

- Lots of potential for future research!
- Other solution concepts
  - Stackelberg equilibrium, correlated equilibrium, (weak) saddles, …
  - tournament games: minimal covering set
- (More) efficient algorithms
  - extend MIP approach to more general game classes
  - continuous payoff sets
  - tractable special cases

Thank you!